Fourier Transformation of Distributions

By

Yoshifumi Ito

Professor Emeritus, The University of Tokushima
209-15 Kamifukuman Hachiman-cho
Tokushima 770-8073, Japan
e-mail address : *yoshifumi@md.pikara.ne.jp*

(Received September 30, 2016)

Abstract

In this paper, we study the Fourier transformation $\mathcal{F}$ of functions in $\mathcal{D}$ and distributions in $\mathcal{D}'$. Thereby we prove the structure theorems of the Fourier images $\mathcal{F}\mathcal{D}$ and $\mathcal{F}\mathcal{D}'$.

2010 Mathematics Subject Classification : Primary 46Fxx.
Key words and phrases : Fourier transformation, distribution, structure theorem of Fourier image, Paley-Wiener type theorem.

Introduction

In this paper, we study the Fourier transformation $\mathcal{F}$ of functions in $\mathcal{D}$ and distributions in $\mathcal{D}'$ on the space $\boldsymbol{R}^d$. Here we assume $d \geq 1$. Thereby we obtain the structure theorems of the Fourier images $\mathcal{F}\mathcal{D}$ and $\mathcal{F}\mathcal{D}'$ by virtue of the Paley-Wiener type theorems. These theorems are the main theorems in this paper. Here we give the new type of Fourier transformation of tempered distributions and distributions. Because the concept of distributions is a generalized concept of classical functions, we define the Fourier transformation of distributions as in the same direction as the Fourier transformation of classical functions. These are very new results. As for the details of these results, we refer to Ito [2], chapters 6 & 7.

Here I show my heartfelt gratitude to my wife Mutuko for her help of typesetting this manuscript.

1 Spaces of functions and distributions

In this section, we give the definitions of several types of functions and distributions on $\boldsymbol{R}^d$.

1.1 Spaces $\mathcal{D}$ and $\mathcal{D}'$

In this subsection, we define the space of functions $\mathcal{D} = \mathcal{D}(\boldsymbol{R}^d)$ and the space of distributions $\mathcal{D}' = \mathcal{D}'(\boldsymbol{R}^d)$.

Assume that the space $C_0^\infty = C_0^\infty(\boldsymbol{R}^d)$ is the vector space of all complex-valued C^∞-functions with compact support on $\boldsymbol{R}^d$. Assume that $\boldsymbol{N} = \{0,\ 1,\ 2,\ \cdots\}$ is the set of natural numbers. We say that $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_d) \in \boldsymbol{N}^d$ is a multi-index.

Then we say that a sequence of functions $\{f_n\}$ in C_0^∞ converges to $f \in C_0^\infty$ if the following conditions (i) and (ii) are satisfied:

(i) There exists some compact set so that its includes all supports of $f_n,\ (n \geq 1)$.

(ii) For any $\alpha \in \boldsymbol{N}^d$, we have

$$\sup_{x \in \boldsymbol{R}^d} |D^\alpha (f_n(x) - f(x))| \to 0,\ (n \to \infty).$$

Here, for $f \in C_0^\infty$, we denote

$$D^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} f.$$

When we define the concept of convergence in C_0^∞ as above, we denote C_0^∞ as $\mathcal{D} = \mathcal{D}(\boldsymbol{R}^d)$. Then $\mathcal{D}$ is a topological vector space.

Assume that, for a compact set K in $\boldsymbol{R}^d$, $\mathcal{D}_K$ is the vector space of all functions in $\mathcal{D}$ such that their supports are contained in K. Then $\mathcal{D}_K$ is a F-space.

Now, we choose an exhausting sequence of compact sets $\{K_j\}$ of $\boldsymbol{R}^d$. Namely, the sequence of compact sets $\{K_j\}$ satisfies the following conditions (i) and (ii):

(i) We have $K_1 \subset K_2 \subset \cdots \subset \boldsymbol{R}^d$ and $\boldsymbol{R}^d = \bigcup_{j=1}^{\infty} K_j$.

(ii) We have $K_j = \mathrm{cl}(\mathrm{int}(K_j))$ and $K_j \subset \mathrm{int}(K_{j+1})$ for $j = 1,\ 2,\ 3,\ \cdots$.

When we denote the strong inductive limit of the inductive system $\{\mathcal{D}_{K_j}\}$ of F-spaces as

$$\varinjlim \mathcal{D}_{K_j},$$

we have the isomorphism

$$\mathcal{D} \cong \varinjlim \mathcal{D}_{K_j}.$$

Here the inclusion mapping $\mathcal{D}_{K_j} \to \mathcal{D}_{K_{j+1}}$ is a compact mapping. Therefore, $\mathcal{D}$ is a DFS-space.

We define that a continuous linear functional $T:\ \mathcal{D} \to \boldsymbol{C}$ is a distribution. We also say simply that T is a distribution.

When $f_n \to f$ in $\mathcal{D}$, we have

$$T(f_n) \to T(f).$$

Now we denote the vector space of all distributions on $\boldsymbol{R}^d$ as $\mathcal{D}' = \mathcal{D}'(\boldsymbol{R}^d)$.

We define a sequence of distributions $\{T_n\}$ converges to $T \in \mathcal{D}'$ if, for any $f \in \mathcal{D}$, we have the equality

$$\lim_{n\to\infty} T_n(f) = T(f).$$

By virtue of this concept of convergence, $\mathcal{D}'$ is a complete TVS. Thereby we define the topology of weak convergence in $\mathcal{D}'$. Namely, this topology of weak convergence is the topology of pointwise convergence. The topology of $\mathcal{D}'$ thus defined coincides with the topology of the strong dual space of $\mathcal{D}$. Here, the topology of the strong dual space of $\mathcal{D}$ is the topology of the uniform convergence on every bounded set in $\mathcal{D}$. This is the topology of strong convergence.

Then, when we choose an exhausting sequence of compact sets $\{K_j\}$ in $\boldsymbol{R}^d$ as above, we define the projective limit $\varprojlim(\mathcal{D}_{K_j})'$ of the projective system of DF-spaces $\{(\mathcal{D}_{K_j})'\}$. Thus we have the isomorphism

$$\mathcal{D}' \cong \varprojlim(\mathcal{D}_{K_j})'$$

as TVS's. Then, because the restriction mapping $(\mathcal{D}_{K_{j+1}})' \to (\mathcal{D}_{K_j})'$ is a compact mapping, $\mathcal{D}'$ is a FS-space.

As for the definitions of the inductive limit and the projective limit of TVS's, we refer to Ito [1] "Theory of Hyperfunctions, I".

Now, for $g \in L^1_{\text{loc}}$, we define the linear functional T_g on $\mathcal{D}$ by the relation

$$T_g(f) = \int g(x)f(x)dx,\ (f \in \mathcal{D}).$$

Then T_g is a distribution on $\boldsymbol{R}^d$ and the correspondence $g \to T_g$ is one to one .

Namely, we have the following theorem.

Theorem 1.1.1(du Bois-Reymond's Lemma) *Assume that Ω is an arbitrary open set in $\boldsymbol{R}^d$ and let $g \in L^1_{\text{loc}}$. Then, if the condition*

$$\int g(x)f(x)dx = 0$$

is satisfied for any $f \in \mathcal{D}(\Omega)$, we have $g(x) = 0$, $(\text{a.e.} x \in \Omega)$.

We say that Theorem 1.1.1 is the **fundamental lemma of the variational problem**.

In such a sense, we identify T_g with g and denote T_g as g.

Then, we have the following theorem.

Theorem 1.1.2 *If the sequence of functions $\{g_n\}$ in L^1_{loc} converges to a function $g \in L^1_{\text{loc}}$ in the sense of L^1_{loc}-topology, g_n also converges to g in the sense of the topology of $\mathcal{D}'$.*

In general, we have the following corollary.

Corollary 1.1.1 *Assume $1 \leq p \leq \infty$. If the sequence of functions $\{g_n\}$ in L^p_{loc} converges to a function $g \in L^p_{\text{loc}}$ in the sense of L^p_{loc}-convergence, g_n also converges to g in the sense of the topology of $\mathcal{D}'$.*

1.2 Spaces $\mathcal{S}$ and $\mathcal{S}'$

In this subsection, we define the space of functions $\mathcal{S}$ and the space of distributions $\mathcal{S}'$.

A function $\varphi(x)$ on $\boldsymbol{R}^d$ is said to be a **rapidly decreasing C^∞-function** if $\varphi(x)$ is a C^∞ -function, and, for any $\alpha,\ \beta \in \boldsymbol{N}^d$, there exists a certain positive constant C such that we have the condition

$$|x^\alpha D^\beta \varphi(x)| \leq C,\ (x \in \boldsymbol{R}^d).$$

Here we put

$$x = {}^t(x_1,\ x_2,\ \cdots,\ x_d),\ |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$$

and

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$$

for $\alpha = (\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_d) \in \boldsymbol{N}^d$.

We have $\varphi \in \mathcal{S}$ if and only if we have the condition

$$\lim_{|x| \to \infty} |x^\alpha D^\beta \varphi(x)| = 0$$

for any $\alpha,\ \beta \in \boldsymbol{N}^d$.

We define a seminorm $p_{\alpha,\,\beta}$ of $\mathcal{S}$ by the relation

$$p_{\alpha,\,\beta}(\varphi) = \sup_x |x^\alpha D^\beta \varphi(x)|$$

for any $\alpha,\ \beta \in \boldsymbol{N}^d$. Then $\mathcal{S}$ is a Fréchet space by virtue of the system of seminorms $\{p_{\alpha,\,\beta};\ \alpha,\ \beta \in \boldsymbol{N}^d\}$ of $\mathcal{S}$.

We define that a sequence of functions $\{\varphi_n\}$ in $\mathcal{S}$ converges to a function φ in $\mathcal{S}$ in the topology of $\mathcal{S}$ if we have the condition

$$p_{\alpha,\,\beta}(\varphi_n - \varphi) \to 0,\ (n \to \infty)$$

for any $\alpha,\ \beta \in \boldsymbol{N}^d$.

We say that a continuous linear functional $T:\ \mathcal{S} \to \boldsymbol{C}$ on $\mathcal{S}$ is a **tempered distribution** in $\mathcal{S}'$.

Since we have the inclusion relation

$$\mathcal{D} \subset \mathcal{S},$$

we have the inclusion relation

$$\mathcal{S}' \subset \mathcal{D}'.$$

Namely the set of all tempered distributions is the special class of distributions.

We define that a sequence of distributions $\{T_n\}$ in $\mathcal{S}'$ converges to a distribution T in $\mathcal{S}'$ in the topology of $\mathcal{S}'$ if we have the condition

$$T_n(\varphi) \to T(\varphi),\ (\varphi \in \mathcal{S}).$$

Since we have the inclusion relation

$$\mathcal{S}' \subset \mathcal{D}',$$

a partial derivative of a distribution in $\mathcal{S}'$ in the sense of distribution in $\mathcal{S}'$ is the same as its partial derivative defined as a distribution in $\mathcal{D}'$

1.3 Spaces $\mathcal{E}$ and $\mathcal{E}'$

In this subsection, we define the space of functions $\mathcal{E}$ and the space of distributions $\mathcal{E}'$.

We denote the vector space of all C^∞-functions on $\boldsymbol{R}^d$ as $\mathcal{E} = C^\infty(\boldsymbol{R}^d)$.

Let $\varphi \in \mathcal{E}$. For an arbitrary compact set K in $\boldsymbol{R}^d$ and any $\alpha \in \boldsymbol{N}^d$, we define a seminorm $p_{K,\,\alpha}$ of $\mathcal{E}$ by the relation

$$p_{K,\,\alpha}(\varphi) = \sup_{x \in K} |D^\alpha \varphi(x)|.$$

Then the function space $\mathcal{E}$ is a Fréchet space with respect to the topology defined by the system of seminorms $\{p_{K,\,\alpha}; K$ is an arbitrary compact set in $\boldsymbol{R}^d$ and any $\alpha \in \boldsymbol{N}^d\}$.

We define that a sequence of functions $\{\varphi_n\}$ in $\mathcal{E}$ converges to a function φ in $\mathcal{E}$ in the topology of $\mathcal{E}$ if we have the condition

$$p_{K,\,\alpha}(\varphi_n - \varphi) \to 0,\ (n \to \infty).$$

for an arbitrary compact set K in $\boldsymbol{R}^d$ and any $\alpha \in \boldsymbol{N}^d$.

Since we have $\mathcal{S} \subset \mathcal{E}$, a continuous linear functional $T : \mathcal{E} \to \boldsymbol{C}$ on $\mathcal{E}$ is considered to be a continuous linear functional on $\mathcal{S}$. Hence we have $T \in \mathcal{S}'$ and T is a tempered distribution. Therefore we have the inclusion relations

$$\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'.$$

Assume $T \in \mathcal{D}'$. Then we define that the closed set F in $\boldsymbol{R}^d$ is the **support** of T when it is the smallest closed set such that we have the condition

$$< T,\ \varphi >= 0,\ (\varphi \in \mathcal{D}(F^c)).$$

Then we denote the support of T as $\mathrm{supp}(T)$. We have $T \in \mathcal{E}'$ if and only if we have $T \in \mathcal{S}'$ and $\mathrm{supp}(T)$ is a compact set. Further, this is equivalent to the condition that we have $T \in \mathcal{D}'$ and $\mathrm{supp}(T)$ is a compact set.

We define a sequence of distributions $\{T_n\}$ in $\mathcal{E}'$ converges to a distribution T in $\mathcal{E}'$ in the topology of $\mathcal{E}'$ if we have the condition

$$T_n(\varphi) \to T(\varphi),\ (\varphi \in \mathcal{E}).$$

Further, since we have $\mathcal{E}' \subset \mathcal{D}'$, a partial derivative of a distribution of $\mathcal{E}'$ in the sense of distributions in $\mathcal{E}'$ is as the same as its partial derivative defined as a distribution in $\mathcal{D}'$.

2 Fourier transformation

In this section, we define the Fourier transformations of several types of functions and distributions for the preparation of the main results in the section 3.

2.1 Fourier transformation of functions in $\mathcal{S}$

In this subsection, we define the Fourier transformation of functions in $\mathcal{S}$ and study their properties. We put $\mathcal{S} = \mathcal{S}(\boldsymbol{R}^d)$. We define the Fourier transformation of $\varphi \in \mathcal{S}$ by the relation

$$\mathcal{F}\varphi(p) = \frac{1}{(\sqrt{2\pi})^d} \int \varphi(x) e^{-ipx} dx, \ (p \in \boldsymbol{R}^d).$$

Here we use the usual notation as follows:

$$x = {}^t(x_1,\ x_2,\ \cdots,\ x_d),\ p = {}^t(p_1,\ p_2,\ \cdots,\ p_d),$$
$$px = p_1x_1 + p_2x_2 + \cdots + p_dx_d,$$
$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2},\ |p| = \sqrt{p_1^2 + p_2^2 + \cdots + p_d^2}.$$

When we denote $\mathcal{F}\varphi = \hat{\varphi}$, we have $\hat{\varphi} \in \mathcal{S}$.
Then we have the following theorem.

Theorem 2.1.1 *For $\alpha \in \boldsymbol{N}^d$ and $\varphi \in \mathcal{S}$, we have the following* (1) *and* (2):

(1) $\mathcal{F}\left((-ix)^\alpha \varphi \right) = D_p^\alpha \mathcal{F}\varphi(p)$ *holds.*

(2) $\mathcal{F}\left(D_x^\alpha \varphi \right) = (ip)^\alpha \mathcal{F}\varphi(p)$ *holds.*

For $\varphi \in \mathcal{S}$, we define the Fourier inverse transformation by the relation

$$(\mathcal{F}^{-1}\varphi)(x) = \frac{1}{(\sqrt{2\pi})^d} \int \varphi(p) e^{ipx} dp, \ (x \in \boldsymbol{R}^d).$$

We put $\mathcal{F}^* = \mathcal{F}^{-1}$ and call it to be the **dual Fourier transformation** or the **Fourier inverse transformation**.
Here we denote

$$\mathcal{F}(\mathcal{S}) = \mathcal{F}\mathcal{S} = \{\hat{\varphi};\ \varphi \in \mathcal{S}\},$$
$$\mathcal{F}^*(\mathcal{S}) = \mathcal{F}^{-1}(\mathcal{S}) = \{\mathcal{F}^{-1}\varphi;\ \varphi \in \mathcal{S}\}.$$

Then we have the following.

Corollary 2.1.1 *For $\varphi \in \mathcal{S}$, we have the equalities*

$$\mathcal{F}^{-1}\mathcal{F}\varphi = \varphi,\ \mathcal{F}\mathcal{F}^{-1}\varphi = \varphi.$$

Therefore, $\mathcal{F}:\ \mathcal{S} \to \mathcal{S}$ is a topological isomorphism.

Corollary 2.1.2 *We have the topological isomorphisms*

$$\mathcal{F}(\mathcal{S}) \cong \mathcal{S},\ \mathcal{F}^*(\mathcal{S}) = \mathcal{F}^{-1}(\mathcal{S}) \cong \mathcal{S}.$$

2.2 Fourier transformation of distributions in $\mathcal{S}'$

In this subsection, we study the Fourier transformation of distributions in $\mathcal{S}'$.

Let $\mathcal{S}' = \mathcal{S}'(\boldsymbol{R}^d)$ be the space of tempered distributions on $\boldsymbol{R}^d$.

Now assume $T \in \mathcal{S}'$. Then, for $\varphi \in \mathcal{S}$, we have $\mathcal{F}^{-1}\varphi \in \mathcal{S}$. Therefore we can define a continuous linear functional

$$S : \varphi \to < T, \mathcal{F}^{-1}\varphi >, \ (\varphi \in \mathcal{S})$$

and we have $S \in \mathcal{S}'$. Namely, we have the equality

$$< S, \varphi > = < T, \mathcal{F}^{-1}\varphi >, \ (\varphi \in \mathcal{S}).$$

Then we say that S is the Fourier transform of T and denote it as $S = \mathcal{F}T$.

This is the new definition of the Fourier transformation of $\mathcal{S}'$. Since a Schwartz distribution is a generalized concept of classical functions. So that, we had better to define the Fourier transformation of Schwartz distributions as in the same direction as the Fourier transformation of classical functions. Thus we define the new type of Fourier transformation of Schwartz distributions.

Namely, for the Fourier transform $\mathcal{F}T \in \mathcal{S}'$ of $T \in \mathcal{S}'$, we have the equality

$$< \mathcal{F}T, \mathcal{F}\varphi > = < T, \varphi >, \ (\varphi \in \mathcal{S}).$$

This is a generalization of Parseval's formula for L^2-functions.

Then the Fourier transformation $\mathcal{F}$ of distributions in $\mathcal{S}'$ is an automorphism of $\mathcal{S}'$ onto $\mathcal{S}'$. Therefore, we have the isomorphism

$$\mathcal{F}\mathcal{S}' \cong \mathcal{S}'.$$

Now we denote the dual mapping of the Fourier transformation $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ as $\mathcal{F}^* : \mathcal{S}' \to \mathcal{S}'$. Then we have the equality

$$\mathcal{F}^*\mathcal{F} = \text{the identity mapping of } \mathcal{S}'.$$

Namely we have the equality

$$\mathcal{F}^{-1} = \mathcal{F}^*.$$

Because we have $\mathcal{E}' \subset \mathcal{S}'$, we remark that the Fourier transformation of distributions in $\mathcal{E}'$ is the same as the Fourier transformation of $\mathcal{S}'$.

Then we have the following theorem.

Theorem 2.2.1 *For $\alpha \in \boldsymbol{N}^d$ and $T \in \mathcal{S}'$, we have the following* (1) *and* (2):

(1) $\mathcal{F}\Big((-ix)^\alpha T \Big) = D_p^\alpha(\mathcal{F}T)$ *holds.*

(2) $\mathcal{F}(D_x^\alpha T) = (ip)^\alpha(\mathcal{F}T)$ *holds.*

2.3 Fourier transformation of functions in $\mathcal{D}$

In this subsection, we study the Fourier transformation of functions in $\mathcal{D}$.

Because we have the inclusion relation $\mathcal{D} \subset \mathcal{S}$, we define the Fourier transformation of functions in $\mathcal{D}$ by restricting the Fourier transformation of functions in $\mathcal{S}$ to $\mathcal{D}$. Namely, for $\varphi \in \mathcal{D}$, we define the Fourier transform $\mathcal{F}\varphi$ of φ by the relation

$$\mathcal{F}\varphi(p) = \frac{1}{(\sqrt{2\pi})^d} \int \varphi(x) e^{-ipx} dx, \ (p \in \boldsymbol{R}^d).$$

Then, because the Fourier transformation

$$\mathcal{F} : \ \mathcal{S} \to \mathcal{S}$$

is a topological isomorphism, we have the commutative diagram:

$$\begin{array}{ccccc} \mathcal{F} : & \mathcal{S} & \to & \mathcal{S} \\ & \cup & & \cup \\ \mathcal{F} : & \mathcal{D} & \to & \mathcal{F}\mathcal{D}. \end{array}$$

Because $\mathcal{D}$ is a closed subspace of $\mathcal{S}$ by virtue of the topology of $\mathcal{S}$, the Fourier transformation

$$\mathcal{F} : \ \mathcal{D} \to \mathcal{F}\mathcal{D}$$

is a topological isomorphism.

Then we have the following theorem.

Theorem 2.3.1 *For $\alpha \in \boldsymbol{N}^d$ and $\varphi \in \mathcal{D}$. Then we have the following* (1) *and* (2):

(1) $\mathcal{F}\left((-ix)^\alpha \varphi \right) = D_p^\alpha (\mathcal{F}\varphi)(p)$ *holds,*

(2) $\mathcal{F}(D_x^\alpha \varphi) = (ip)^\alpha (\mathcal{F}\varphi)(p)$ *holds.*

2.4 Fourier transformation of distributions in $\mathcal{D}'$

In this subsection, we study the Fourier transformation of distributions in $\mathcal{D}'$.

Assume that $\mathcal{D}' = \mathcal{D}'(\boldsymbol{R}^d)$ is the space of Schwartz distributions on $\boldsymbol{R}^d$. Now assume $T \in \mathcal{D}'$. Then, because $\mathcal{F}^{-1}\varphi \in \mathcal{D}$ holds for $\varphi \in \mathcal{F}\mathcal{D}$, we can define a continuous linear functional

$$S : \ \varphi \to < T, \ \mathcal{F}^{-1}\varphi >, \ (\varphi \in \mathcal{F}\mathcal{D})$$

and we have $S \in (\mathcal{FD})'$. Namely, we have the equality

$$< S,\ \varphi >=< T,\ \mathcal{F}^{-1}\varphi >,\ (\varphi \in \mathcal{FD}).$$

Then we say that S is a Fourier transform of T and denote it as $S = \mathcal{F}T$. This is the new definition of the Fourier transformation of distributions of $\mathcal{D}'$ as in the case of distributions in $\mathcal{S}'$. Namely, the Fourier transform $\mathcal{F}T \in \mathcal{FD}' \cong (\mathcal{FD})'$ of $T \in \mathcal{D}'$ satisfies the relation

$$< \mathcal{F}T,\ \mathcal{F}\varphi >=< T,\ \varphi >,\ (\varphi \in \mathcal{D}).$$

This is a generalization of Parseval's formula as in the case of $\mathcal{S}'$.

Then the Fourier transformation $\mathcal{F}$ of distributions in $\mathcal{D}'$ is the topological isomorphism of $\mathcal{D}'$ onto $\mathcal{FD}'$.

Therefore, if we denote the dual mapping of the Fourier transformation $\mathcal{F}:\ \mathcal{D} \to \mathcal{FD}$ as $\mathcal{F}^*:\ (\mathcal{FD})' \to \mathcal{D}'$, we have the equality

$$\mathcal{F}^*\mathcal{F} = \text{the identity mapping of}\, \mathcal{D}'.$$

Then we have the following theorem.

Theorem 2.4.1 *For $\alpha \in \boldsymbol{N}^d$ and $T \in \mathcal{D}'$, we have the following* (1) *and* (2):

(1) $\mathcal{F}\Big(\,(-ix)^\alpha T\,\Big) = D_p^\alpha(\mathcal{F}T)$ *holds.*

(2) $\mathcal{F}(D_x^\alpha T) = (ip)^\alpha(\mathcal{F}T)$ *holds.*

3 Paley-Wiener type theorems and structure theorems

In this section, we prove the Paley-Wiener type theorems for functions in $\mathcal{D}$ and distributions in $\mathcal{D}'$. Thereby we prove the structure theorems of $\mathcal{FD}$ and $\mathcal{FD}'$. These are the main results of this paper.

3.1 Fourier image of the space $\mathcal{D}$

In this subsection, we prove the Parlay-Wiener type theorem for $\mathcal{D}$ and the structure theorem of the Fourier image $\mathcal{FD}$.

We denote a point ζ on $\boldsymbol{C}^d$ as follows:

$$\zeta = {}^t(\zeta_1,\ \zeta_2,\ \cdots,\ \zeta_d),$$

$$\zeta_j = \xi_j + i\eta_j,\ (\xi_j,\ \eta_j \in \boldsymbol{R}^d,\ j = 1,\ 2,\ \cdots,\ d),$$

$$\operatorname{Im}\zeta = {}^{\mathrm{t}}(\operatorname{Im}\zeta_1,\ \operatorname{Im}\zeta_2,\ \cdots,\ \operatorname{Im}\zeta_{\mathrm{d}}) = {}^{\mathrm{t}}(\eta_1,\ \eta_2,\ \cdots,\ \eta_{\mathrm{d}}).$$

Then a function $F(\zeta)$ on $\boldsymbol{C}^d$ means a function $F(\zeta) = F(\zeta_1,\ \zeta_2,\ \cdots,\ \zeta_d)$.
Further, for $x = {}^t(x_1,\ x_2,\ \cdots,\ x_d) \in \boldsymbol{R}^d$, we denote

$$\zeta x = \zeta_1 x_1 + \zeta_2 x_2 + \cdots + \zeta_d x_d.$$

Then we have the following Paley-Wiener type theorem.

Theorem 3.1.1 (Paley-Wiener type theorem) *Let B be a certain positive constant. Then the following conditions (1) and (2) are equivalent:*

(1) *An entire function $F(\zeta)$ on $\boldsymbol{C}^d$ satisfies the condition that, for an arbitrary natural number N, there exists a certain positive constant C_N such that we have the inequality*

$$|F(\zeta)| \leq C_N(1 + |\zeta|)^{-N} e^{B|\operatorname{Im}\zeta|},\ (\zeta \in \boldsymbol{C}^d).$$

(2) *A function $F(\zeta)$ is equal to the Fourier-Laplace transform*

$$F(\zeta) = \frac{1}{(\sqrt{2\pi})^d} \int \varphi(x) e^{-i\zeta x} dx,\ (\zeta \in \boldsymbol{C}^d)$$

of a certain function $\varphi \in \mathcal{D}$ which satisfies the condition $\operatorname{supp}(\varphi) \subset \{|x| \leq B\}$.

Now we put

$$K_j = \{x \in R^d;\ |x| \leq j\},\ (j = 1,\ 2,\ 3,\ \cdots).$$

Then the sequence of compact sets $\{K_j\}$ is an exhausting sequence of compact sets in $\boldsymbol{R}^d$. Therefore we have the isomorphism

$$\mathcal{D} \cong \varinjlim \mathcal{D}_{K_j}$$

as TVS's. Here $\varinjlim \mathcal{D}_{K_j}$ denotes the strong inductive limit of the inductive system of F-spaces $\{\mathcal{D}_{K_j}\}$. Therefore, by virtue of the Fourier transformation $\mathcal{F}$ of $\mathcal{D}$, we can define the Fourier transform $\mathcal{F}\varphi$ of a function φ in each $\mathcal{D}_{K_j}$.
Then we denote

$$\mathcal{F}\mathcal{D} = \hat{\mathcal{D}},\ \mathcal{F}\mathcal{D}_{K_j} = (\mathcal{D}_{K_j})\hat{},\ (j = 1,\ 2,\ 3,\ \cdots).$$

Further we have the inclusion relations

$$\mathcal{D}_{K_1} \subset \mathcal{D}_{K_2} \subset \cdots \subset \mathcal{D}_{K_j} \subset \cdots ,$$

$$(\mathcal{D}_{K_1})\hat{} \subset (\mathcal{D}_{K_2})\hat{} \subset \cdots \subset (\mathcal{D}_{K_j})\hat{} \subset \cdots .$$

Here, we remark that , for every $j \geq 1$, a function of $(\mathcal{D}_{K_j})\hat{}$ is characterized by Theorem 3.1.1.

Then we have the following theorem.

Theorem 3.1.2 *We use the notation in the above. Then we have the following isomorphisms* (1) $\sim$ (4):

(1) $\mathcal{D} \cong \varinjlim_j \mathcal{D}_{K_j}$.

(2) $\hat{\mathcal{D}} \cong \varinjlim_j (\mathcal{D}_{K_j})\hat{}$.

(3) $\mathcal{D}_{K_j} \cong (\mathcal{D}_{K_j})\hat{}$, $(j = 1,\ 2,\ 3,\ \cdots)$.

(4) $\mathcal{D} \cong \hat{\mathcal{D}}$.

3.2 Fourier image of the space $\mathcal{D}'$

In this subsection, we prove the Paley-Wiener type theorem for $\mathcal{D}'$ and the structure theorem of the Fourier image $\mathcal{F}\mathcal{D}'$.

When the support of $T \in \mathcal{S}'$ is included in the compact set $K_B = \{|x| \leq B\}$, we can consider that $T \in (\mathcal{D}_{K_B})'$ holds. Therefore, this means that we have $T \in \mathcal{D}'$ such that its support is included in the compact set K_B. Thus we have the following theorem.

Theorem 3.2.1 (Paley-Wiener type theorem) *Assume that B is a certain positive constant. Then the following conditions* (1) *and* (2) *are equivalent*:

(1) *An entire function $F(\zeta)$ on $\boldsymbol{C}^d$ satisfies the condition*

$$|F(\zeta)| \leq C(1 + |\zeta|)^N e^{B|\mathrm{Im}\,\zeta|},\ (\zeta \in \boldsymbol{C}^d)$$

for a certain positive constants C and N.

(2) *A function* $F(\zeta)$ *is equal to the Fourier-Laplace transform*

$$F(\zeta) = \frac{1}{(\sqrt{2\pi})^d} < T_x,\ e^{-i\zeta x} >,\ (\zeta \in \boldsymbol{C}^d)$$

of a certain distribution $T \in \mathcal{D}'$ *which satisfies the condition*

$$\mathrm{supp}(T) \subset K_B = \{|x| \leq B\}.$$

Thus we remark that, for every $j \geq 1$, a distribution of $\mathcal{F}(\mathcal{D}_{K_j})' \cong (\mathcal{F}\mathcal{D}_{K_j})'$ is characterized by Theorem 3.2.1.

Then we have the following theorem.

Theorem 3.2.2 *We use the same notation as in Theorem* 3.1.2. *Then we have the following isomorphisms* (1) $\sim$ (4):

(1) $\mathcal{D}' \cong \varprojlim_j (\mathcal{D}_{K_j})'$.

(2) $\mathcal{F}\mathcal{D}' \cong \varprojlim_j \mathcal{F}(\mathcal{D}_{K_j})'$.

(3) $(\mathcal{F}\mathcal{D})' \cong \varprojlim_j (\mathcal{F}\mathcal{D}_{K_j})'$.

(4) $\mathcal{D}' \cong \mathcal{F}\mathcal{D}' \cong (\mathcal{F}\mathcal{D})'$.

In Theorem 3.2.2, the symbol $\varprojlim_j (\mathcal{D}_{K_j})'$ denotes the projective limit of the projective system of DF-spaces $\{(\mathcal{D}_{K_j})'\}$.

References

[1] Y.Ito, *Theory of Hyperfunctions*, I, preprint, 2010.8.29, (in Japanese).

[2] ———, *Fourier Analysis*, preprint, 2014.8.25, (in Japanese).