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Asymptotic Behavior of Solutions for Kirchhoff Type Dissipative Wave Equations in Unbounded Domains

By

Kosuke Ono

Department of Mathematical Sciences,
Tokushima University, Tokushima 770-8502, JAPAN
e-mail : *k.ono@tokushima-u.ac.jp*
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Abstract

Consider the Cauchy problem for the non-degenerate Kirchhoff type dissipative wave equations with the initial data belonging to $H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ in unbounded domains. When the coefficient ρ or the initial energy $E(0)$ is small at least, we show the global existence theorem and derive decay estimates of energies in the L^2-frame. Moreover, when the initial data belong to $L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$ in addition, we improve the decay rates of the solutions.

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1 Introduction

In this paper we consider the Cauchy problem for the non-degenerate Kirchhoff type dissipative wave equations :

$$\begin{cases} \rho u'' + \left(1 + \int_{\mathbb{R}^N} |A^{1/2}u(\cdot,t)|^2 dx\right)^{\gamma} Au + u' = 0 \qquad \text{in} \quad \mathbb{R}^N \times [0,\infty)\,, \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) \qquad \text{in} \quad \mathbb{R}^N\,, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is an unknown real value function, $' = \partial/\partial t$, $A = -\Delta = -\sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator with domain $\mathcal{D}(A) = H^2(\mathbb{R}^N)$, $\rho > 0$ and $\gamma > 0$ are positive constants.

Equations (1.1) describes small amplitude vibrations of an elastic string when the dimension N is one (see Kirchhoff [9] for the original equation, and also see Carrier [5], Dickey [6]). Equations including non-local terms like (1.1) are called Kirchhoff type.

When the initial data belong to Sobolev spaces, Arosio and Garavaldi [1] have carried out detailed analysis about the existence of local solutions for the Kirchhoff type equations (also see [2], [4], [6], [22], and the references cited therein).

Yamada [21] and Brito [3] studied on the global solvability in suitable Sobolev spaces using the energy method. Moreover, Yamada [21] derived some decay estimates of the solutions like (1.10) in the L^2-frame when $\gamma \geq 1$ (see Hashimoto and Yamazaki [7] for abstract cases). In previous paper [15], we improved the decay rates in [21] and also derived the decay estimates (1.10)–(1.11) when $\gamma \geq 1$ and the initial data $[u_0, u_1] \in H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ are small (see [17] for bounded domain cases).

On the other hand, in addition to the energy method in the L^2-frame, using the Fourier transform method in the $L^1 \cap L^2$-frame, we can improve the decay rates in (1.10)–(1.11) and in this paper we obtain the better decay estimates (1.12)–(1.14) when any $\gamma > 0$. Moreover, under the assumption that the coefficient $\rho > 0$ or the initial energy $E(0)$ is small at least, we will show the global solvability for (1.1).

We define the energies $E(t)$ and $H(t)$ by

$$E(t) \equiv \rho \|u'(t)\|^2 + P_M(t) \tag{1.2}$$

and

$$H(t) \equiv \rho \frac{\|A^{1/2}u'(t)\|^2}{(1+M(t))^\gamma} + \|Au(t)\|^2 \tag{1.3}$$

where $\|\cdot\|$ is the usual norm in $L^2 = L^2(\mathbb{R}^N)$ and

$$M(t) \equiv \|A^{1/2}u(t)\|^2 \tag{1.4}$$

and

$$P_M(t) \equiv \int_0^{M(t)} (1+\mu)^\gamma \, d\mu = \frac{1}{\gamma+1}\left((1+M(t))^{\gamma+1} - 1\right). \tag{1.5}$$

Then, it is easy to see that

$$M(t) \leq P_M(t) \leq (1+M(t))^\gamma \, M(t), \tag{1.6}$$

and in particular, when $t = 0$ we have

$$E(0) \leq \rho \|u_1\|^2 + \left(1 + \|A^{1/2}u_0\|^2\right)^\gamma \|A^{1/2}u_0\|^2 \tag{1.7}$$

and

$$H(0) \leq \rho \|A^{1/2}u_1\|^2 + \|Au_0\|^2. \tag{1.8}$$

Our main result is as follows.

Theorem 1.1 *Let the initial data* $[u_0, u_1]$ *belong to* $H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. *Suppose that the coefficient* $\rho > 0$ *and the initial data* $[u_0, u_1]$ *satisfy*

$$\rho E(0)\left(\gamma^2 H(0)\right) < 1 . \tag{1.9}$$

Then the problem (1.1) *admits a unique global solution* $u(t)$ *in the class* $C^0([0,\infty); H^2(\mathbb{R}^N)) \cap C^1([0,\infty); H^1(\mathbb{R}^N)) \cap C^2([0,\infty); L^2(\mathbb{R}^N))$ *satisfying*

$$\|A^{1/2}u(t)\|^2 \le C(1+t)^{-1}, \quad \|u'(t)\|^2 + \|Au(t)\|^2 \le C(1+t)^{-2}, \tag{1.10}$$

$$\|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \le C(1+t)^{-3} \qquad \text{for} \quad t \ge 0 . \tag{1.11}$$

Moreover, if the initial data $[u_0, u_1]$ *belong to* $L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$ *in addition, the solution* $u(t)$ *satisfies*

$$\|u(t)\|^2 \le C(1+t)^{-\eta} \qquad \text{with} \quad \eta = \min\left\{\frac{N}{2}, 2\right\}, \tag{1.12}$$

$$\|A^{1/2}u(t)\|^2 \le C(1+t)^{-1-\eta}, \qquad \|u'(t)\|^2 + \|Au(t)\|^2 \le C(1+t)^{-2-\eta}, \tag{1.13}$$

$$\|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \le C(1+t)^{-3-\eta} \qquad \text{for} \quad t \ge 0 , \tag{1.14}$$

where C *is some positive constant.*

Theorem 1.1 follows from Theorem 2.3, Theorem 3.6, and Theorem 4.5 in the continuing sections.

The notations we use in this paper are standard. The symbol $(\cdot,\cdot)$ means the inner product in $L^2 = L^2(\mathbb{R}^N)$ or sometimes duality between the space X and its deal X'. The symbol $\|\cdot\|_{L^p}$ means the norm in $L^p = L^p(\mathbb{R}^N)$ (we often denote $\|\cdot\| = \|\cdot\|_{L^2}$). Positive constants will be denoted by C and will change from line to line.

2 Existence

We obtain the following local existence theorem by standard arguments and we omit the proof here (see [1], [14], [18], [19], [20], and the references cited therein).

Proposition 2.1 *Suppose that the initial data* $[u_0, u_1]$ *belong to* $H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. *Then the problem* (1.1) *admits a unique local solution* $u(t)$ *in the class* $C^0([0,T); H^2(\mathbb{R}^N)) \cap C^1([0,T); H^1(\mathbb{R}^N)) \cap C^2([0,T); L^2(\mathbb{R}^N))$ *for some* $T = T(\|u_0\|_{H^2}, \|u_1\|_{H^1}) > 0$. *Moreover,* $\|u_0\|_{H^2} + \|u_1\|_{H^1} < \infty$ *for* $t \ge 0$, *then we can take that* $T = \infty$.

Proposition 2.2 *The solution* $u(t)$ *of* (1.1) *satisfies*

$$E(t) + 2\int_0^t \|u'(s)\|^2\, ds = E(0) \tag{2.1}$$

and

$$\|u(t)\|^2 \le J(0) \qquad \text{with} \quad J(0) \equiv 2(2\|u_0\|^2 + 3\rho E(0))\,. \tag{2.2}$$

Proof. Multiplying (1.1) by $2u'(t)$ and integrating it over $\mathbb{R}^N$, we have

$$\frac{d}{dt}E(t) + 2\|u'(t)\|^2 = 0\,, \tag{2.3}$$

and integrating (2.3) in time t, we obtain the energy identity (2.1).

Multiplying (1.1) by $2u(t)$ and integrating it over $\mathbb{R}^N$, we have

$$\frac{d}{dt}\|u(t)\|^2 + 2(1+M(t))^\gamma M(t) = 2\rho\left(\|u'(t)\|^2 - \frac{d}{dt}(u'(t),u(t))\right)\,, \tag{2.4}$$

and integrating (2.4) in time t, we observe from the Young inequality that

$$\begin{aligned}
&\|u(t)\|^2 + 2\int_0^t (1+M(s))^\gamma M(s)\,ds \\
&= \|u_0\|^2 + 2\rho\left((u_0,u_1) - (u(t),u'(t)) + \int_0^t \|u'(s)\|^2\,ds\right) \\
&\le \|u_0\|^2 + \left(\|u_0\|^2 + \rho^2\|u_1\|^2\right) + \left(\frac{1}{2}\|u(t)\|^2 + 2\rho^2\|u'(t)\|^2\right) + 2\rho\int_0^t \|u'(s)\|^2 ds
\end{aligned}$$

and from (2.1) that

$$\frac{1}{2}\|u(t)\|^2 \le 2\|u_0\|^2 + 3\rho E(0)$$

which implies the desired estimate (2.2). □

Theorem 2.3 *Let the initial data $[u_0,u_1]$ belong to $H^2(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Suppose that the coefficient $\rho>0$ and the initial data $[u_0,u_1]$ satisfy*

$$\gamma^2\rho E(0)H(0) < 1\,. \tag{2.5}$$

Then, the problem (1.1) admits a unique global solution $u(t)$ in the class $C^0([0,\infty);H^2(\mathbb{R}^N))\cap C^1([0,\infty);H^1(\mathbb{R}^N))\cap C^2([0,\infty);L^2(\mathbb{R}^N))$ satisfying

$$\|u(t)\|^2 \le J(0) \quad \text{and} \quad M(t)\le E(t)\le E(0) \quad \text{and} \quad H(t)\le H(0) \tag{2.6}$$

(*see* (2.2), (1.7), (1.8) *for* $J(0)$, $E(0)$, $H(0)$, *respectively*).

Proof. Let $u(t)$ be a solution of (1.1) on $[0,T]$. Since $\delta H(0)<1$ with $\delta = \gamma^2\rho E(0)$ by (2.5), putting

$$T_1 \equiv \sup\left\{t\in[0,\infty)\;\middle|\;\delta H(s) < 1 \text{ for } 0\le s<t\right\}\,,$$

we see that $T_1 > 0$. If $T_1 < T$, then

$$\delta H(t) < 1 \quad \text{for} \quad 0 \le t < T_1 \quad \text{and} \quad \delta H(T_1) = 1\,. \tag{2.7}$$

Multiplying (1.1) by $2(1+M(t))^{-\gamma}Au'(t)$ and integrating it over $\mathbb{R}^N$, we have

$$\frac{d}{dt}H(t) + 2\left(1 + \frac{\gamma}{2}\rho\frac{M'(t)}{1+M(t)}\right)\frac{\|A^{1/2}u'(t)\|^2}{(1+M(t))^\gamma} = 0\,.$$

Since it follows from (2.1) and (2.7) that

$$\begin{aligned} 1 + \frac{\gamma}{2}\rho\frac{M'(t)}{1+M(t)} &\ge 1 - \gamma\rho\|u'(t)\|\|Au(t)\| \\ &\ge 1 - \gamma(\rho E(0))^{\frac12}H(t)^{\frac12} = 1 - (\delta H(t))^{\frac12} \ge 0 \end{aligned}$$

for $0 \le t \le T_1$, we have

$$\frac{d}{dt}H(t) \le 0 \quad \text{or} \quad H(t) \le H(0) \tag{2.8}$$

for $0 \le t \le T_1$. Then, we observe from (2.5) and (2.8) that

$$\delta H(t) \le \delta H(0) < 1$$

for $0 \le t \le T_1$ which is a contradiction to (2.7), and hence, we have that $T_1 \ge T$.

Thus, from (2.1), (2.2), and (2.8) we obtain that $\|u(t)\|_{H^2} + \|u'(t)\|_{H^1} \le C$ for $0 \le t \le T$. Therefore, by the second statement of Proposition 2.1, we conclude that the problem (1.1) admits a unique global solution, and also we obtain (2.6). □

3 Decay

In this section we will derive some decay estimates of the solution $u(t)$ of (1.1) given by Theorem 2.3. The following generalized Nakao type inequality is useful to derive decay estimates of energies (see [8], [12], [16] for the proof, and also [11], [13]).

Lemma 3.1 *Let $\phi(t)$ be a non-negative function on $[0,\infty)$ and satisfy*

$$\sup_{t\le s\le t+1} \phi(s)^{1+\alpha} \le \big(k_0\phi(t)^\alpha + k_1(1+t)^{-\beta}\big)\big(\phi(t) - \phi(t+1)\big) + k_2(1+t)^{-\gamma}$$

with certain constants $k_0,\ k_1,\ k_2 \ge 0$, $\alpha > 0$, $\beta > -1$, and $\gamma > 0$. Then, the function $\phi(t)$ satisfies

$$\phi(t) \le C_0(1+t)^{-\theta}\,, \qquad \theta = \min\left\{\frac{1+\beta}{\alpha}\,,\ \frac{\gamma}{1+\alpha}\right\}$$

for $t \ge 0$ with some constant C_0 depending on $\phi(0)$.

Proposition 3.2 *Under the assumption of Theorem 2.3, it holds that*

$$M(t) \le E(t) \le C(1+t)^{-1}\,. \tag{3.1}$$

Proof. Integrating (2.3) over $[t, t+1]$, we have

$$2\int_t^{t+1} \|u'(s)\|^2\,ds = E(t) - E(t+1) \qquad (\equiv 2D(t)^2\,)\,. \tag{3.2}$$

Then there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|u'(t_j)\|^2 \le 4D(t)^2 \qquad \text{for} \quad j = 1, 2\,. \tag{3.3}$$

On the other hand, since it follows from (1.2) and (2.4) that

$$\begin{aligned} &E(t) + (1 + M(t)^\gamma)M(t) - P_M(t) \\ &\qquad = 2\rho\|u'(t)\|^2 - \rho\frac{d}{dt}(u'(t), u(t)) - (u'(t), u(t))\,, \end{aligned} \tag{3.4}$$

integrating (3.4) over $[t_1, t_2]$, we observe from (1.6), (3.2), and (3.3) that

$$\begin{aligned} &\int_{t_1}^{t_2} E(s)\,ds \\ &\quad \le \int_{t_1}^{t_2} \left(2\rho\|u'(s)\|^2 - \rho\frac{d}{dt}(u'(s), u(s)) - (u'(s), u(s))\right) ds \\ &\quad \le 2\rho\int_t^{t+1} \|u'(s)\|^2\,ds + \rho\sum_{j=1}^{2}\|u'(t_j)\|\|u(t_j)\| + \int_t^{t+1}\|u'(s)\|\|u(s)\|\,ds \\ &\quad \le 2\rho D(t)^2 + CD(t)\sup_{t\le s\le t+1} g(s) \qquad \text{with} \quad g(t)^2 \equiv \|u(t)\|^2\,. \end{aligned} \tag{3.5}$$

Integrating (2.3) over $[t, t_2]$, we have from (3.2) and (3.5) that

$$\begin{aligned} E(t) &\le E(t_2) + 2\int_t^{t_2}\|u'(s)\|^2\,ds \\ &\le 2\int_{t_1}^{t_2} E(s)\,ds + 2\int_t^{t+1}\|u'(s)\|^2\,ds \\ &\le CD(t)^2 + CD(t)\sup_{t\le s\le t+1} g(s)\,. \end{aligned}$$

Since $2D(t)^2 = E(t) - E(t+1) \le E(t)$ by (3.2), we observe

$$\begin{aligned} E(t)^2 &\le C\left(D(t)^2 + \sup_{t\le s\le t+1} g(s)^2\right) D(t)^2 \\ &\le C\left(E(t) + \sup_{t\le s\le t+1} g(s)^2\right)(E(t) - E(t+1))\,. \end{aligned} \tag{3.6}$$

Thus, since $E(t) \le E(0)$ and $g(t) \equiv \|u(t)\|^2 \le J(0)$ by (2.1) and (2.2), we observe

$$E(t)^2 \le C\left(E(t) - E(t+1)\right)\,, \tag{3.7}$$

and hence, applying Lemma 3.1 to (3.7), we obtain the desired estimate (3.1). □

Proposition 3.3 *Under the assumption of Theorem 2.3, it holds that*

$$F(t) \equiv \rho\|A^{1/2}u'(t)\|^2 + (1+M(t))^\gamma \|Au(t)\|^2 \le C(1+t)^{-2}\,. \tag{3.8}$$

Proof. Multiplying (1.1) by $2Au'(t)$ and integrating it over $\mathbb{R}^N$, we have from (2.6) that

$$\begin{aligned}\frac{d}{dt}F(t) + 2\|A^{1/2}u'(t)\|^2 &= \gamma(1+M(t))^{\gamma-1}M'(t)\|Au(t)\|^2 \\ &\le C\|A^{1/2}u(t)\|\|A^{1/2}u'(t)\|\|Au(t)\|^2\,,\end{aligned} \tag{3.9}$$

and the Young inequality yields

$$\frac{d}{dt}F(t) + \|A^{1/2}u'(t)\|^2 \le Cf(t)^2 \qquad \text{with} \quad f(t)^2 \equiv M(t)\|Au(t)\|^4\,. \tag{3.10}$$

Integrating (3.10) over $[t, t+1]$, we have

$$\int_t^{t+1} \|A^{1/2}u'(s)\|^2\,ds = F(t) - F(t+1) + C \sup_{t\le s\le t+1} f(s)^2 \qquad (\equiv D(t)^2\,)\,. \tag{3.11}$$

Then, there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|A^{1/2}u'(t_j)\|^2 \le 4D(t)^2 \qquad \text{for} \quad j = 1, 2\,. \tag{3.12}$$

On the other hand, multiplying (1.1) by $Au(t)$ and integrating it over $\mathbb{R}^N$, we have

$$\begin{aligned}&(1+M(t))^\gamma\|Au(t)\|^2 \\ &\quad = \rho\left(\|A^{1/2}u'(t)\|^2 - \frac{d}{dt}(A^{1/2}u'(t), A^{1/2}u(t))\right) - (A^{1/2}u'(t), A^{1/2}u(t))\end{aligned}$$

or

$$F(t) = 2\rho\|A^{1/2}u'(t)\|^2 - \rho\frac{d}{dt}(A^{1/2}u'(t), A^{1/2}u(t)) - (A^{1/2}u'(t), A^{1/2}u(t))\,, \tag{3.13}$$

and integrating (3.13) over $[t_1, t_2]$, we have from (3.11) and (3.12) that

$$\begin{aligned}
&\int_{t_1}^{t_2} F(s)\,ds \\
&\quad \le 2\rho \int_t^{t+1} \|A^{1/2}u'(s)\|^2\,ds + \rho \sum_{j=1}^{2} \|A^{1/2}u'(t_j)\|\|A^{1/2}u(t_j)\| \\
&\qquad + \int_t^{t+1} \|A^{1/2}u'(s)\|\|A^{1/2}u(s)\|\,ds \\
&\quad \le 2\rho D(t)^2 + CD(t) \sup_{t\le s\le t+1} g(s) \qquad \text{with} \quad g(t)^2 \equiv M(t)\,. \qquad (3.14)
\end{aligned}$$

Moreover, there exists $t_* \in [t_1, t_2]$ such that

$$F(t_*) \le 2\int_{t_1}^{t_2} F(s)\,ds\,. \qquad (3.15)$$

For $\tau \in [t, t+1]$, integrating (3.9) over $[\tau, t_*]$ (or $[t_*, \tau]$), we have from (3.10) and (3.15) that

$$\begin{aligned}
F(\tau) &= F(t_*) + \int_\tau^{t_*} \left(2\|A^{1/2}u'(s)\|^2 - \gamma(1+M(s))^{\gamma-1}M'(s)\|Au(s)\|^2\right) ds \\
&\le 2\int_{t_1}^{t_2} F(s)\,ds + \int_t^{t+1} \left(C\|A^{1/2}u'(s)\|^2 + Cf(s)^2\right) ds
\end{aligned}$$

and from (3.11) and (3.14) that

$$\sup_{t\le s\le t+1} F(s) \le CD(t)^2 + CD(t) \sup_{t\le s\le t+1} g(s) + C \sup_{t\le s\le t+1} f(s)^2\,.$$

Since $D(t)^2 = F(t) - F(t+1) + C \sup_{t\le s\le t+1} f(s)^2 \le F(t) + C \sup_{t\le s\le t+1} f(s)^2$ by (3.11), we observe

$$\begin{aligned}
&\sup_{t\le s\le t+1} F(s)^2 \le C\left(D(t)^2 + \sup_{t\le s\le t+1} g(s)^2\right) D(t)^2 + C \sup_{t\le s\le t+1} f(s)^4 \\
&\quad \le C\left(F(t)^2 + \sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right)(F(t) - F(t+1)) \\
&\qquad + CF(t) \sup_{t\le s\le t+1} f(s)^2 + C\left(\sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right) \sup_{t\le s\le t+1} f(s)^2
\end{aligned}$$

or

$$\begin{aligned}
\sup_{t\le s\le t+1} F(s)^2 &\le C\left(F(t)^2 + \sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right)(F(t) - F(t+1)) \\
&\quad + C\left(\sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right) \sup_{t\le s\le t+1} f(s)^2\,. \qquad (3.16)
\end{aligned}$$

Since it follows from (3.10), (2.6), and (3.1) that

$$f(t)^2 \equiv M(t)\|Au(t)\|^2 \leq \begin{cases} C(1+t)^{-1}\,, \\ C(1+t)^{-1}F(t)\,, \end{cases} \tag{3.17}$$

and from (3.13) and (3.1) that

$$g(t)^2 \equiv M(t) \leq C(1+t)^{-1}\,, \tag{3.18}$$

we have

$$\sup_{t\leq s\leq t+1} F(s)^2 \leq C\left(F(t)+(1+t)^{-1}\right)(F(t)-F(t+1)) + C(1+t)^{-2}\sup_{t\leq s\leq t+1} F(s)$$

or

$$\sup_{t\leq s\leq t+1} F(s)^2 \leq C\left(F(t)+(1+t)^{-1}\right)(F(t)-F(t+1)) + C(1+t)^{-4}\,. \tag{3.19}$$

Thus, applying Lemma 3.1 to (3.19), we obtain the desired estimate (3.8). □

Proposition 3.4 *Under the assumption of Theorem 2.3, it holds that*

$$\|u'(t)\|^2 \leq C(1+t)^{-2}\,. \tag{3.20}$$

Proof. Multiplying (1.1) by $2u'(t)$ and integrating it over $\mathbb{R}^N$, we have

$$\rho\frac{d}{dt}\|u'(t)\|^2 + 2\|u'(t)\|^2 = -2(1+M(t))^{\gamma}(Au(t),u'(t))\,,$$

and using the Young inequality we observe from (2.6) and (3.8) that

$$\rho\frac{d}{dt}\|u'(t)\|^2 + \|u'(t)\|^2 \leq h(t)^2 \tag{3.21}$$

with

$$h(t)^2 \equiv (1+M(t))^{2\gamma}\|Au(t)\|^2 \leq C(1+t)^{-2} \tag{3.22}$$

which gives the desired estimate (3.20). □

Proposition 3.5 *Under the assumption of Theorem 2.3, it holds that*

$$\begin{aligned} L(t) &\equiv \rho\|u''(t)\|^2 + (1+M(t))^{\gamma}\,\|A^{1/2}u'(t)\|^2 + \frac{\gamma}{2}\,(1+M(t))^{\gamma-1}\,|M'(t)|^2 \\ &\leq C(1+t)^{-3}\,. \end{aligned} \tag{3.23}$$

Proof. Multiplying (1.1) differentiated with respect to t by $2u''(t)$ and integrating it over $\mathbb{R}^N$, we have

$$\frac{d}{dt}L(t) + 2\|u''(t)\|^2 \tag{3.24}$$
$$= 3\gamma(1+M(t))^{\gamma-1}M'(t)\|A^{1/2}u'(t)\|^2 + \frac{\gamma(\gamma-1)}{2}(1+M(t))^{\gamma-2}(M'(t))^3$$
$$\le Cf(t)^2 \qquad \text{with} \quad f(t)^2 \equiv \|u'(t)\|\|Au(t)\|\|A^{1/2}u'(t)\|^2\,. \tag{3.25}$$

Integrating (3.25) over $[t, t+1]$, we have

$$2\int_t^{t+1} \|u''(s)\|^2\,ds \le L(t) - L(t+1) + C\sup_{t\le s\le t+1} f(s)^2 \quad (\equiv 2D(t)^2). \tag{3.26}$$

Then, there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\|u''(t_j)\|^2 \le 4D(t)^2 \qquad \text{for} \quad j = 1, 2\,. \tag{3.27}$$

On the other hand, multiplying (1.1) differentiated with respect to t by $u'(t)$ and integrating it over $\mathbb{R}^N$, we have

$$(1+M(t))^\gamma\|A^{1/2}u'(t)\|^2 + \frac{\gamma}{2}(1+M(t))^{\gamma-1}|M'(t)|^2$$
$$= \rho\left(\|u''(t)\|^2 - \frac{d}{dt}(u''(t), u'(t))\right) - (u''(t), u'(t))$$

or

$$L(t) = 2\rho\|u''(t)\|^2 - \rho\frac{d}{dt}(u''(t), u'(t)) - (u''(t), u'(t))\,, \tag{3.28}$$

and integrating (3.28) over $[t_1, t_2]$, we observe from (3.26) and (3.27) that

$$\int_{t_1}^{t_2} L(s)\,ds$$
$$\le 2\rho\int_t^{t+1}\|u''(s)\|^2\,ds + \rho\sum_{j=1}^{2}\|u''(t_j)\|\|u'(t_j)\| + \int_t^{t+1}\|u''(s)\|\|u'(s)\|\,ds$$
$$\le 2\rho D(t)^2 + CD(t)\sup_{t\le s\le t+1} g(s) \qquad \text{with} \quad g(t)^2 \equiv \|u'(t)\|^2\,. \tag{3.29}$$

Moreover, there exists $t_* \in [t_1, t_2]$ such that

$$L(t_*) \le 2\int_{t_1}^{t_2} L(s)\,ds\,. \tag{3.30}$$

For $\tau \in [t, t+1]$, integrating (3.24) over $[\tau, t_*]$ (or $[t_*, \tau]$), we have from (3.25) and (3.30) that

$$\begin{aligned} L(\tau) &= L(t_*) + \int_\tau^{t_*} \Bigg(2\|u''(s)\|^2 - 3\gamma(1+M(s))^{\gamma-1} M'(s) \|A^{1/2}u'(s)\|^2 \\ &\qquad + \frac{\gamma(\gamma-1)}{2}(1+M(s))^{\gamma-2}(M'(s))^3 \Bigg)\, ds \\ &\le 2\int_{t_1}^{t_2} L(s)\, ds + \int_t^{t+1} \left(C\|u''(s)\|^2 + Cf(s)^2 \right) ds \end{aligned}$$

and from (3.26) and (3.29) that

$$\sup_{t \le s \le t+1} L(s) \le CD(t)^2 + CD(t) \sup_{t \le s \le t+1} g(s) + C \sup_{t \le s \le t+1} f(s)^2$$

or

$$\sup_{t \le s \le t+1} L(s)^2 \le C \left(D(t)^2 + \sup_{t \le s \le t+1} g(s)^2 \right) D(t)^2 + C \sup_{t \le s \le t+1} f(s)^4 .$$

Since $2D(t)^2 = L(t) - L(t+1) + C \sup_{t \le s \le t+1} f(s)^2 \le L(t) + C \sup_{t \le s \le t+1} f(s)^2$ by (3.26), we observe

$$\begin{aligned} \sup_{t \le s \le t+1} L(s)^2 &\le C \left(L(t) + \sup_{t \le s \le t+1} f(s)^2 + \sup_{t \le s \le t+1} g(s)^2 \right) (L(t) - L(t+1)) \\ &\quad + CL(t) \sup_{t \le s \le t+1} f(s)^2 + C \left(\sup_{t \le s \le t+1} f(s)^2 + \sup_{t \le s \le t+1} g(s)^2 \right) \sup_{t \le s \le t+1} f(s)^2 \end{aligned}$$

or

$$\begin{aligned} \sup_{t \le s \le t+1} L(s)^2 &\le C \left(L(t) + \sup_{t \le s \le t+1} f(s)^2 + \sup_{t \le s \le t+1} g(s)^2 \right) (L(t) - L(t+1)) \\ &\quad + C \left(\sup_{t \le s \le t+1} f(s)^2 + \sup_{t \le s \le t+1} g(s)^2 \right) \sup_{t \le s \le t+1} f(s)^2 . \end{aligned} \tag{3.31}$$

Since it follows from (3.25), (3.8), and (3.20) that

$$f(t)^2 \equiv \|u'(t)\| \|Au(t)\| \|A^{1/2}u'(t)\|^2 \le \begin{cases} C(1+t)^{-4} , \\ C(1+t)^{-2} L(t) , \end{cases} \tag{3.32}$$

and from (3.28) and (3.20) that

$$g(t)^2 \equiv \|u'(t)\|^2 \le C(1+t)^{-2} , \tag{3.33}$$

we have

$$\sup_{t \le s \le t+1} L(s)^2 \le C \left(L(t) + (1+t)^{-2} \right) (L(t) - L(t+1)) + C(1+t)^{-4} \sup_{t \le s \le t+1} L(s)$$

or

$$\sup_{t\le s\le t+1} L(s)^2 \le C\left(L(t)+(1+t)^{-2}\right)(L(t)-L(t+1))+C(1+t)^{-8}\,. \quad (3.34)$$

Thus, applying Lemma 3.1 to (3.34), we obtain the desired estimate (3.23). □

Gathering Propositions 3.2–3.5, we conclude the following theorem.

Theorem 3.6 *Suppose that the assumption of Theorem 2.3 is fulfilled. Then, the solution* $u(t)$ *of* (1.1) *satisfies*

$$\|A^{1/2}u(t)\|^2 \le C(1+t)^{-1}\,, \quad (3.35)$$

$$\|u'(t)\|^2+\|Au(t)\|^2 \le C(1+t)^{-2}\,, \quad (3.36)$$

$$\|A^{1/2}u'(t)\|^2+\|u''(t)\|^2 \le C(1+t)^{-3} \qquad \text{for} \quad t\ge 0\,, \quad (3.37)$$

where C *is some positive constant.*

Proof. (3.35) follows from (3.1). (3.36) follows from (3.8) and (3.20). (3.37) follows from (3.23). □

4 Improved Decay

Under the additional condition that the initial data $[u_0,u_1]$ belong to $L^1(\mathbb{R}^N)$ $\times L^1(\mathbb{R}^N)$, we will improve the decay rates (3.35)–(3.37) given by Theorem 3.6. In order to achieve our purpose, first we need to derive the decay estimate of L^2-norm of the solution $u(t)$.

We denote the Fourier transform of $g(x)$ by

$$\mathcal{F}(g(x))(\xi) \equiv \hat{g}(\xi) \equiv (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-i\xi\cdot x} g(x)\,dx\,,$$

where $\xi\cdot x=\sum_{j=1}^N \xi_j x_j$.

Through the Fourier transform, we can rewrite (1.1) to the following equation :

$$\begin{cases} \rho\hat{u}''+\hat{u}'+|\xi|^2\hat{u}=f(M(t))\widehat{Au} & \text{in} \quad \mathbb{R}^N_\xi\times[0,\infty)\,, \\ \hat{u}(\xi,0)=\widehat{u_0}(\xi) \quad \text{and} \quad \hat{u}'(\xi,0)=\widehat{u_1}(\xi) & \text{in} \quad \mathbb{R}^N_\xi\,, \end{cases} \quad (4.1)$$

where $f(M)=1-(1+M)^\gamma$. Then, we obtain the integral form for (4.1) :

$$\hat{u}(\xi,t)=\widehat{u_L}(\xi,t)+\widehat{u_N}(\xi,t) \quad (4.2)$$

where

$$\widehat{u_L}(\xi,t)=\frac{1}{2}\left(\phi_1(\xi,t)+\phi_2(\xi,t)\right)\widehat{u_0}(\xi)+\phi_2(\xi,t)\widehat{u_1}(\xi)\,, \quad (4.3)$$

$$\widehat{u_N}(\xi,t)=\int_0^t \phi_2(\xi,t-s)f(M(s))\widehat{Au}(\xi,s)\,ds\,, \quad (4.4)$$

and we set

$$\phi_1(\xi,t)=\begin{cases}2e^{-\frac{t}{2\rho}}\cosh\frac{\lambda t}{2\rho} & \text{if} \quad |\xi|<1/(2\sqrt{\rho})\,,\\ 2e^{-\frac{t}{2\rho}}\cos\frac{\sigma t}{2\rho} & \text{if} \quad |\xi|\ge 1/(2\sqrt{\rho})\,,\end{cases}$$

$$\phi_2(\xi,t)=\begin{cases}2e^{-\frac{t}{2\rho}}\frac{1}{\lambda}\sinh\frac{\lambda t}{2\rho} & \text{if} \quad |\xi|<1/(2\sqrt{\rho})\,,\\ 2e^{-\frac{t}{2\rho}}\frac{1}{\sigma}\sin\frac{\sigma t}{2\rho} & \text{if} \quad |\xi|\ge 1/(2\sqrt{\rho})\,,\end{cases}$$

and $\lambda=\sqrt{1-4\rho|\xi|^2}$ and $\sigma=\sqrt{4\rho|\xi|^2-1}$.

Proposition 4.1 *Under the assumption of Theorem 2.3, if the initial data* $[u_0,u_1]$ *belong to* $L^1(\mathbb{R}^N)\times L^1(\mathbb{R}^N)$, *it holds that*

$$\|u(t)\|^2\le C(1+t)^{-\eta} \qquad \textit{with} \quad \eta=\min\left\{\frac{N}{2}\,,\,2\right\}\,. \tag{4.5}$$

Proof. By the standard argument for the linear dissipative wave equation (see Matsumura [10] and Kawashima et al. [8] for the proof), concerning the linear part (4.3) in the integral form (4.2), we have

$$\|u_L(t)\|^2\le C(1+t)^{-\frac{N}{4}}\left(\|u_0\|+\|u_1\|+\|u_0\|_{L^1}+\|u_1\|_{L^1}\right)\,. \tag{4.6}$$

Next, in order to estimate the nonlinear part (4.4) in the integral form (4.2), we set that for $j=1,2,3,4$,

$$\chi_j(\xi)=\begin{cases}1 & \text{if} \quad \xi\in X_j\,,\\ 0 & \text{if} \quad \xi\notin X_j\,,\end{cases}$$

where

$$X_1\equiv\left\{\xi\;\middle|\;|\xi|<1/(4\sqrt{\rho})\right\}\,,\quad X_2\equiv\left\{\xi\;\middle|\;1/(4\sqrt{\rho})\le|\xi|<1/(2\sqrt{\rho})\right\}\,,$$
$$X_3\equiv\left\{\xi\;\middle|\;1/(2\sqrt{\rho})\le|\xi|<1/\sqrt{\rho}\right\}\,,\quad X_4\equiv\left\{\xi\;\middle|\;1/\sqrt{\rho}\le|\xi|\right\}\,.$$

Using the Parseval identity together with (4.4), we observe

$$\|u_N(t)\|\le\int_0^t\|\phi_2(\xi,t-s)\widehat{Au}(\xi,s)\||f(M(s))|\,ds$$

and

$$\begin{aligned}&\|\phi_2(\xi,t-s)\widehat{Au}(\xi,s)\|\\ &\quad\le\|\chi_1(\xi)\phi_2(\xi,t-s)|\xi|^2\hat{u}(\xi,s)\|+\sum_{j=2}^{4}\|\chi_j(\xi)\phi_2(\xi,t-s)\widehat{Au}(\xi,s)\|\\ &\quad\le C\sup_{\xi\in X_1}|\xi|^2|\phi_2(\xi,t-s)|\|u(s)\|+C\sum_{j=2}^{4}\sup_{\xi\in X_j}|\phi_2(\xi,t-s)|\|Au(s)\|\,.\end{aligned}$$

(a) When $\xi \in X_1$, since $\sqrt{3}/2 < \lambda \leq 1$ and $(-1+\lambda)/(2\rho) \leq -2|\xi|^2$, we have

$$\sup_{\xi \in X_1} |\xi|^2 |\phi_2(\xi,t)| \leq C \sup_{\xi \in X_1} |\xi|^2 e^{-2|\xi|^2 t} \leq C(1+t)^{-1}\,.$$

(b) When $\xi \in X_2$, since $0 < \lambda \leq \sqrt{3}/2$, we have

$$\begin{aligned}\sup_{\xi \in X_2} |\phi_2(\xi,t)| &\leq Cte^{-\frac{t}{2\rho}} \sup_{\xi \in X_2} \frac{2\rho}{\lambda t} \left| \int_0^1 \frac{d}{d\theta} \left(\sinh \frac{\lambda t}{2\rho}\theta \right) d\theta \right| \\ &\leq Cte^{-\frac{t}{2\rho}} \sup_{\xi \in X_2} \left| \int_0^1 \cosh \frac{\lambda t}{2\rho}\theta\, d\theta \right| \leq Cte^{-(1-\frac{\sqrt{3}}{2})\frac{t}{2\rho}}\,.\end{aligned}$$

(c) When $\xi \in X_3$, since $0 \leq \sigma < \sqrt{3}$, we have

$$\begin{aligned}\sup_{\xi \in X_3} |\phi_2(\xi,t)| &\leq Cte^{-\frac{t}{2\rho}} \sup_{\xi \in X_3} \frac{2\rho}{\sigma t} \left| \int_0^1 \frac{d}{d\theta} \left(\sin \frac{\sigma t}{2\rho}\theta \right) d\theta \right| \\ &\leq Cte^{-\frac{t}{2\rho}} \sup_{\xi \in X_3} \left| \int_0^1 \cos \frac{\sigma t}{2\rho}\theta\, d\theta \right| \leq Cte^{-\frac{t}{2\rho}}\,.\end{aligned}$$

(d) When $\xi \in X_4$, since $\sigma \geq \sqrt{2}$, we have

$$\sup_{\xi \in X_4} |\phi_2(\xi,t)| \leq Ce^{-\frac{t}{2\rho}} \sup_{\xi \in X_4} \frac{1}{\sigma} \left| \sin \frac{\sigma t}{2\rho} \right| \leq Ce^{-\frac{t}{2\rho}}\,.$$

Thus, we obtain

$$\begin{aligned}\|u_N(t)\| &\leq C \int_0^t (1+t-s)^{-1} |f(M(s))| \|u(s)\|\, ds \\ &\quad + C \int_0^t e^{-\delta(t-s)} |f(M(s))| \|Au(s)\|\, ds \end{aligned} \tag{4.7}$$

with some $\delta > 0$.

Therefore, the estimates (4.2), (4.6), and (4.7) yield

$$\begin{aligned}\|u(t)\| \leq C(1+t)^{-\frac{N}{4}} &+ C \int_0^t (1+t-s)^{-1} |f(M(s))| \|u(s)\|\, ds \\ &+ C \int_0^t e^{-\delta(t-s)} |f(M(s))| \|Au(s)\|\, ds\,.\end{aligned}$$

On the other hand, since it follows from (3.35) that

$$|f(M(t))| = |(1+M(t))^\gamma - 1| \leq CM(t) \leq C(1+t)^{-1}\,,$$

we observe from (3.36) that

$$\begin{aligned}\|u(t)\| &\le C(1+t)^{-\frac{N}{4}} + C\int_0^t (1+t-s)^{-1}(1+s)^{-1}\|u(s)\|\,ds \\ &\quad + C\int_0^t e^{-\delta(t-s)}(1+s)^{-3}\,ds \\ &\le C(1+t)^{-\min\{\frac{N}{4},3\}} + C\int_0^t (1+t-s)^{-1}(1+s)^{-1}\|u(s)\|\,ds\end{aligned}$$

and since $\|u(t)\|$ is bounded (see (2.6)) we have

$$\|u(t)\| \le C(1+t)^{-\min\{\frac{N}{4},1\}}$$

which implies the desired estimate (4.5). □

Proposition 4.2 *Under the assumption of Proposition 4.1, it holds that*

$$M(t) \le E(t) \le C(1+t)^{-\eta} \qquad \textit{with} \quad \eta = \min\left\{\frac{N}{2}\,,\,2\right\}. \tag{4.8}$$

Proof. We derive (4.8) by the same way as in the proof of Proposition 3.2. Instead of (2.6), we use

$$g(t)^2 \equiv \|u(t)\|^2 \le C(1+t)^{-\eta}\,,$$

and we observe from (3.6) that

$$E(t) \le C\left(E(t) + (1+t)^{-\eta}\right)\left(E(t) - E(t+1)\right). \tag{4.9}$$

Thus, applying Lemma 3.1 to (4.9), we obtain the desired estimate (4.8). □

Proposition 4.3 *Under the assumption of Proposition 4.1, it holds that*

$$F(t) \equiv \rho\|A^{1/2}u'(t)\|^2 + (1+M(t))^{\gamma}\,\|Au(t)\|^2 \le C(1+t)^{-2-\eta} \tag{4.10}$$

and

$$\|u'(t)\|^2 \le C(1+t)^{-2-\eta} \qquad \textit{with} \quad \eta = \min\left\{\frac{N}{2}\,,\,2\right\}. \tag{4.11}$$

Proof. We derive (4.10) by the same way as in the proof of Proposition 3.3. Instead of (3.17) and (3.18), we use

$$f(t)^2 \equiv M(t)\|Au(t)\|^4 \le \begin{cases} C(1+t)^{-1-\eta}\,, \\ C(1+t)^{-1-\eta}F(t)\,, \end{cases}$$

and

$$g(t)^2 \equiv M(t) \le C(1+t)^{-1-\eta}\,,$$

and we observe from (3.16) that

$$\sup_{t\le s\le t+1} F(s)^2 \le C\left(F(t)+(1+t)^{-1-\eta}\right)(F(t)-F(t+1)) \\ + C(1+t)^{-2-2\eta}\sup_{t\le s\le t+1} F(s)\,. \tag{4.12}$$

Thus, applying Lemma 3.1 together with the Young inequality to (4.12), we obtain the desired estimate (4.10).

Moreover, we derive (4.11) by the same way as in the proof of Proposition 3.4. Instead of (3.22), we use

$$h(t)^2 \equiv (1+M(t))^{2\gamma}\|Au(t)\|^2 \le C(1+t)^{-2-\eta}\,,$$

and we observe from (3.21) that

$$\rho\frac{d}{dt}\|u'(t)\|^2 + \|u'(t)\|^2 \le C(1+t)^{-2-\eta}$$

which gives the desired estimate (4.11). □

Proposition 4.4 *Under the assumption of Proposition 4.1, it holds that*

$$L(t) \equiv \rho\|u''(t)\|^2 + (1+M(t))^{\gamma}\|Au'(t)\|^2 + \frac{\gamma}{2}(1+M(t))^{\gamma-1}|M'(t)|^2 \\ \le C(1+t)^{-3-\eta}\,. \tag{4.13}$$

Proof. We derive (4.13) by the same way as in the proof of Proposition 3.5. Instead of (3.32) and (3.33), we use

$$f(t)^2 \equiv \|u'(t)\|\|Au(t)\|\|A^{1/2}u'(t)\|^2 \le \begin{cases} C(1+t)^{-4-2\eta}\,, \\ C(1+t)^{-2-\eta}L(t)\,, \end{cases}$$

and

$$g(t)^2 \equiv \|u'(t)\|^2 \le C(1+t)^{-2-\eta}\,,$$

and we observe from (3.31) that

$$\sup_{t\le s\le t+1} L(s)^2 \le C\left(L(t)+(1+t)^{-2-\eta}\right)(L(t)-L(t+1)) \\ + C(1+t)^{-4-2\eta}\sup_{t\le s\le t+1} L(s)\,. \tag{4.14}$$

Thus, applying Lemma 3.1 together with the Young inequality to (4.14), we obtain the desired estimate (4.13). □

Gathering Proposition 4.1–4.4, we arrived the following theorem.

Theorem 4.5 *In addition to the assumption of Theorem 2.3, suppose that the initial data* $[u_0, u_1]$ *belong to* $L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$*. Then, the solution* $u(t)$ *of* (1.1) *satisfies*

$$\|u(t)\|^2 \le C(1+t)^{-\eta} \qquad \text{with} \quad \eta = \min\left\{\frac{N}{2}, 2\right\}, \tag{4.15}$$

$$\|A^{1/2}u(t)\|^2 \le C(1+t)^{-1-\eta}, \tag{4.16}$$

$$\|u'(t)\|^2 + \|Au(t)\|^2 \le C(1+t)^{-2-\eta}, \tag{4.17}$$

$$\|A^{1/2}u'(t)\|^2 + \|u''(t)\|^2 \le C(1+t)^{-3-\eta} \qquad \text{for} \quad t \ge 0, \tag{4.18}$$

where C *is some positive constant.*

Proof. (4.15) follows from (4.5). (4.16) follows from (4.8). (4.17) follows from (4.10) and (4.11). (4.18) follows from (4.13). □

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