Local Switching of Some Signed Graphs

By

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Abstract

Some signed graphs are transformed to trees by a sequence of local switchings. We give some examples of such signed graphs to investigate when signed graphs are transformed to trees by local switching.

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Introduction

Local switching of signed graphs is introduced by P. J. Cameron, J.J. Seidel and S. V. Tsaranov in [3]. Some signed Fushimi trees are transformed to trees by a sequence of local switchings [4]. Signed cycles with odd parity are transformed to trees by a sequence of local switchings, but signed cycles with even parity cannot be transformed to trees by any means [5]. What kinds of graphs are transformed to trees by a sequence of local switchings? It is important and interesting to give examples of signed graphs which are transformed to trees by a sequence of local switchings. In this note, we give rather simple examples of such signed graphs.

We state briefly basic facts about signed graphs. A graph $G = (V, E)$ consists of an $n$-set $V$ (the vertices) and a set $E$ of unordered pairs from $V$ (the edges). A signed graph $(G, f)$ is a graph $G$ with a signing $f : E \to \{1, -1\}$ of the edges. We set $E^+ = f^{-1}(+1)$ and $E^- = f^{-1}(-1)$. For any subset $U \subseteq V$ of vertices, let $f_U$ denote the signing obtained from $f$ by reversing the sign of each edge which has one vertex in $U$. This defines on the set of signings an equivalence relation, called switching. The equivalence classes $\{f_U : U \subseteq V\}$ are the signed switching classes of the graph $G = (V, E)$.

Let $i \in V$ be a vertex of $G$, and $V(i)$ be the neighbours of $i$. The local graph of $(G, f)$ at $i$ has $V(i)$ as its vertex set, and as edges all edges $\{j, k\}$ of $G$ for which $f(i, j)f(j, k)f(k, i) = -1$. A rim of $(G, f)$ at $i$ is any union of connected
components of local graph at \(i\). Let \(J\) be any rim at \(i\), and let \(K = V(i) \setminus J\). Local switching of \((G, f)\) with respect to \((i, J)\) is the following operation:

(i) delete all edges of \(G\) between \(J\) and \(K\); (ii) for any \(j \in J, k \in K\) not previously joined, introduce an edge \(\{j, k\}\) with sign chosen so that \(f(i, j) f(j, k) f(k, i) = -1\); (iii) change the signs of all edges from \(i\) to \(J\); (iv) leave all other edges and signs unaltered. Let \(\Omega_n\) be the set of switching classes of signed graphs of order \(n\). Local switching, applied to any vertex and any rim at the vertex, gives a relation on \(\Omega\) which is symmetric but not transitive. The equivalence classes of its transitive closure are called the clusters of order \(n\).

1. Signed graphs which are transformed into trees by local switching

A connected graph \(G = (V, E)\) is called a Fushimi tree if each block of \(G\) is a complete graph. A complete graph is a Fushimi tree of one block. Let \(a\) be a cut vertex of a Fushimi tree \(G\). If \(G\) is divided exactly \(m\) connected components when the cut vertex \(a\) is deleted, in the present paper, we say that the Fushimi degree (simply F-degree) of the cut vertex \(a\) is \(m\). A connected subgraph of a Fushimi tree \(G\) is called a sub-Fushimi tree if it consists of some blocks of \(G\). A block of Fushimi tree is said to be pendant if it has only one cut vertex. It is evident that any Fushimi tree has at least two pendant blocks.

A signed Fushimi tree is called a Fushimi tree with positive sign (or simply a positive Fushimi tree) if we can switch all signs of edges into \(+1\). A tree is always considered as a Fushimi tree with positive sign. A tree with only two leaves is said to be a line tree or simply a line in the present paper.

A \(k\)-cycle \(C^k = (V, E)\), where \(V = \{a_1, a_2, \ldots, a_k\}\), \(E = \{a_1a_2, a_2a_3, \ldots, a_{k-1}a_k, a_k a_1\}\), will be denoted simply \(C^k = a_1a_2 \cdots a_k a_1\). For signed cycles, there are two switching classes, which are distinguished by the parity or the balance, where the parity of a signed cycle is the parity of the number of its edges which carry a positive sign and the balance is the product of the signs on its edges [3]. In the forthcoming paper [5], we will show the following two theorems.

**Theorem 1.** Let \(G\) be a positive Fushimi tree whose any cut vertex has F-degree 2. We can transform \(G\) into a line tree by a sequence of local switchings.

**Theorem 2.** Let \(C^k\) be a \(k\)-cycle. Then, it is transformed to a tree by a sequence of local switchings if and only if its parity is odd.

We will show

**Theorem 3.** Let \(G = (V, E)\) be a signed graph with \(V = \{a_1, a_2, \ldots, a_n, b_2, b_3, \ldots, b_{m-1}\}\) and \(E = \{a_1a_2, a_2a_3, \ldots, a_{n-1}a_n, a_na_1, a_1b_2, b_2b_3, \ldots, b_{m-1}a_n\}\). Consider two cycles \(A^n = a_1a_2 \cdots a_na_1\) and \(B^m = a_1b_2 \cdots b_{m-1}a_n\). Then, the graph is transformed to a tree by a sequence of local switchings if and only if both parities of \(A^n\) and \(B^m\) are odd.
Proof. Assume that the parity of $A^n$ is odd. By a sequence of local switchings, $(a_2, J = \{a_3\})$, $(a_3, J = \{a_4\})$, \ldots, $(a_{n-2}, J = \{a_{n-1}\})$, $(a_3, J = \{a_2\})$, $(a_4, J = \{a_3\})$, \ldots, $(a_{n-2}, J = \{a_{n-3}\})$, $(a_{n-1}, J = \{a_1\})$, we get a signed graph with edge set $E = \{a_2a_3, \ldots, a_{n-2}a_{n-1}, a_1b_2, b_2b_3, \ldots, b_{m-1}a_n, a_na_{n-1}, a_{n-1}a_1\}$. The parity of the cycle $a_1b_2b_3\cdots b_{m-1}a_na_{n-1}a_1$ is odd if and only if the parity of $B^m$ is odd. In this case, this cycle is transformed to a tree by a sequence of local switchings. If the parity of $A^n$ is even, by a sequence of local switchings, $(a_2, J = \{a_3\})$, $(a_3, J = \{a_4\})$, \ldots, $(a_{n-2}, J = \{a_{n-1}\})$, $(a_3, J = \{a_2\})$, $(a_4, J = \{a_3\}, \ldots$, $(a_{n-2}, J = \{a_{n-1}\})$, we get a signed graph with edge set $E = \{a_1a_2a_3, a_{n-2}a_{n-1}a_n, a_{n-1}a_n, a_{n-2}a_{n-1}, a_{n-1}a_n, a_{n-2}a_{n-1}, a_{n-1}a_n, a_{n-2}a_{n-1}, a_{n-1}a_n, a_{n-2}a_{n-1}, a_{n-1}a_n, a_{n-2}a_{n-1}, a_{n-1}a_n\}$. As the sign of the edge $a_{n-1}a_n$ is $-1$, the cycle $a_1a_{n-1}a_na_1$ can not be transformed to a line.

2. Examples of signed graphs which are transformed into trees

For $j = 3, 4, \ldots, 8$, set signed graphs $T_j = (V, E)$ as follows.

$V = \{a_1, a_2, \ldots, a_{j+2}\}$, $E^+ = \{a_ia_{i+1}, a_ia_{i+2}(i = 1, 2, \ldots, j), a_{j+1}a_{j+2}\}$, $E^- = \emptyset$.

Then, we have

Proposition 4. The signed graphs $T_3, T_4, T_5, T_6, T_7$ are transformed to trees by a sequence of local switchings, but $T_8$ can not be transformed to a tree by a sequence of local switchings.

Proof. By a sequence of local switchings, $(a_3, J = \{a_2\})$, $(a_5, J = \{a_3\})$, from $T_3$, we get a tree with edge set $E = \{a_1a_3, a_3a_5, a_4a_5, a_2a_5\}$.

By a sequence of local switchings, $(a_3, J = \{a_2\})$, $(a_5, J = \{a_3, a_6\})$, $(a_6, J = \{a_2\})$, from $T_4$, we get a tree with edge set $E = \{a_1a_3, a_3a_5, a_4a_5, a_5a_6, a_2a_5\}$.

By a sequence of local switchings, $(a_3, J = \{a_2\})$, $(a_5, J = \{a_3, a_6, a_7\})$, $(a_7, J = \{a_2, a_5\})$, $(a_6, J = \{a_2\})$, from $T_5$, we get a tree with edge set $E = \{a_1a_3, a_3a_5, a_5a_7, a_4a_7, a_7a_8, a_2a_5\}$.

By a sequence of local switchings, $(a_3, J = \{a_2\})$, $(a_5, J = \{a_3, a_6, a_7\})$, $(a_7, J = \{a_5\})$, $(a_8, J = \{a_7\})$, $(a_5, J = \{a_2\})$, $(a_3, J = \{a_5\})$, from $T_6$, we get a tree with edge set $E = \{a_1a_3, a_3a_5, a_5a_8, a_8a_2, a_2a_5, a_2a_7, a_7a_4\}$.

By a sequence of local switchings, $(a_3, J = \{a_2\})$, $(a_5, J = \{a_3, a_6, a_7\})$, $(a_7, J = \{a_5, a_8, a_9\})$, $(a_9, J = \{a_7\})$, $(a_2, J = \{a_9\})$, $(a_4, J = \{a_8\})$, $(a_8, J = \{a_9\})$, from $T_7$, we get a tree with edge set $E = \{a_1a_3, a_3a_5, a_5a_7, a_7a_9, a_9a_8, a_8a_4, a_8a_2, a_2a_6\}$.

By a sequence of local switchings, $(a_3, J = \{a_2\})$, $(a_5, J = \{a_3, a_6, a_7\})$, $(a_7, J = \{a_5, a_8\})$, $(a_8, J = \{a_7\})$, $(a_2, J = \{a_7\})$, from $T_8$, we get a signed graph with edge set $E^+ = \{a_1a_3, a_3a_5, a_5a_7, a_7a_9, a_9a_{10}, a_9a_{10}, a_8a_2, a_2a_7, a_2a_6\}$, $E^- = \{a_4a_7\}$.

But this graph can not be transformed to a tree at all.
It is rather difficult to decide that a given signed graph can not be transformed to a tree by a sequence of local switchings. We describe some facts concerning with this point.

**Remark.** A signed cycle with even parity can not be transformed to a tree by a sequence of local switchings. Hence, we do not make a 3-cycle with even parity by local switching. Set \( G_1 = (V,E) \) be a signed graph with vertex set \( V = \{a_1, a_2, b_1, b_2, c\} \) and edge sets \( E^+ = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2, a_1c\} \), \( E^- = \{a_2c\} \). By local switching at \( b_1 \) or \( b_2 \), we get a 3-cycle with even parity, we can not apply it. By local switching at \( a_1 \) or \( a_2 \), if \( b_1 \) is in \( J \) and \( b_2 \) is in \( K \) or if the reverse holds, we get a 3-cycle with even parity. Similarly, set \( G_2 = (V,E) \) be a signed graph with vertex set \( V = \{a_1, a_2, b_1, \ldots, b_n, c\} \) and edge sets \( E^+ = \{a_1b_1, \ldots, a_nb_n, a_1c\} \), \( E^- = \{a_2c\} \). We can not do local switching at any \( b_i \) (\( 1 \leq i \leq n \)). If we apply local switching at \( a_1 \) or \( a_2 \), all \( b_i \)'s must be in \( J \) or in \( K \).

Let \( Q_3 = (V,E) \) be a signed graph with \( V = \{a_1, a_2, \ldots, a_7, a_8\} \), \( E^+ = \{a_1a_2, a_1a_3, a_2a_4, a_3a_5, a_5a_6, a_5a_7, a_7a_8\} \) and \( E^- = \{a_2a_4, a_4a_6, a_6a_8\} \). and \( Q_4 = (V,E) \) be a signed graph with vertex set \( V = \{a_1, a_2, \ldots, a_7, a_8\} \) and edge sets \( E^+ = \{a_1a_2, a_1a_3, a_2a_4, a_3a_5, a_5a_6, a_6a_7, a_7a_8, a_9a_10\} \) and \( E^- = \{a_2a_4, a_4a_6, a_6a_8, a_8a_10\} \).

Now we have

**Proposition 5.** The graph \( Q_3 \) is transformed to the graph \( T_6 \), and hence to a tree, by a sequence of local switchings. The graph \( Q_4 \) is transformed to the graph \( T_8 \) by a sequence of local switchings. Hence this graph can not be transformed to a tree by a sequence of local switchings.

**Proof.** By a sequence of local switchings, \( (a_1, J = \{a_2\}), (a_7, J = \{a_8\}), (a_6, J = \{a_5\}), (a_8, J = \{a_7\}) \), we get \( T_6 \) from \( Q_3 \). Similarly, by a sequence of local switchings, \( (a_1, J = \{a_2\}), (a_{10}, J = \{a_9\}), (a_7, J = \{a_8\}), (a_6, J = \{a_5\}), (a_9, J = \{a_{10}\}), (a_8, J = \{a_7\}), (a_{10}, J = \{a_9\}) \), we get \( T_8 \) from \( Q_4 \).

Define \( QH_2, QH_3 \) and \( QH_4 \) as follows;

- \( QH_2 = (V,E), V = \{a_1, a_2, a_3, a_4, b_1, b_2\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_1b_1, a_3b_2\}, E^- = \{a_4a_1\} \);
- \( QH_3 = (V,E), V = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_1b_1, a_2b_2, a_3b_3\}, E^- = \{a_4a_1\} \);
- \( QH_4 = (V,E), V = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_1b_1, a_2b_2, a_3b_3, a_4b_4\}, E^- = \{a_4a_1\} \);

We can prove

**Proposition 6.** The signed graphs \( QH_2, QH_3, QH_4 \) are transformed to trees by a sequence of local switchings.

**Proof.** By a sequence of local switchings, \( (a_4, J = \{a_1\}), (a_3, J = \{a_1, b_2\}), (b_2, J = \{a_3\}) \), from \( QH_2 \), we get a tree with edge set \( E = \{b_1a_1, a_1a_3, a_3b_2, b_2a_2, a_2a_4\} \). By a sequence of local switchings, \( (a_4, J = \{a_1\}), (a_3, J = \{a_1, b_3\}), (b_3, J = \{a_3\}) \), from \( QH_3 \), we get a tree with edge set \( E = \{b_1a_1, a_1a_3, a_3b_3, b_3a_2, b_2a_4, a_2b_2\} \). By a sequence of local switchings, \( (a_2, J = \{a_3\}), (a_3, J = \{a_3\}) \), from \( QH_4 \), we get a tree with edge set \( E = \{b_1a_1, a_1a_3, a_3b_3, b_3a_2, b_2a_4, a_2b_2\} \).
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{a_2, a_4}), (b_3, J = \{a_3\}), (b_2, J = \{a_4\}), (a_4, J = \{a_3, b_4\}), (b_4, J = \{a_4\}), from QH_4, we get a tree with edge set \( E = \{b_1a_1, a_1a_3, a_3a_4, a_4b_4, b_4a_3, b_3b_2, b_2a_2\} \).

Set signed graphs \( PH_1, PH_2, PH_3(1), PH_3(2), PH_4, PH_5 \) as follows;

\( PH_1 = (V, E), V = \{a_1, a_2, a_3, a_4, a_5, b_1\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1\}, E^- = \emptyset \);

\( PH_2 = (V, E), V = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2\}, E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2, a_3b_3\}, E^- = \emptyset \);

\( PH_3(1) = (V, E), V = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3\}, E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2, a_3b_3\}, E^- = \emptyset \);

\( PH_3(2) = (V, E), V = \{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4\}, E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2, a_3b_3, a_4b_4\}, E^- = \emptyset \);

\( PH_4 = (V, E), V = \{a_1, a_2, a_3, a_4, as, b_1, b_2, b_3\}, E = \{a_1a_2, a_2a_3, asa_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5\}, E^- = \emptyset \);

\( PH_5 = (V, E), V = \{a_1, a_2, a_3, a_4, as, b_1, b_2, b_3, b_4, b_5\}, E = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5\}, E^- = \emptyset \);

Now we obtain

**Proposition 7.** The graphs \( PH_1, PH_2, PH_3(1), PH_3(2), PH_4 \) are transformed to trees, by a sequence of local switchings. The graph \( PH_5 \) can not be transformed to a tree by a sequence of local switchings.

Proof. By a sequence of local switchings, \( (a_2, J = \{a_3\}), (a_3, J = \{a_4\}) \), \( (a_4, J = \{a_5\}) \), \( (a_2, J = \{a_1, a_5\}) \), \( (a_3, J = \{a_4\}) \), from \( PH_1 \), we get a tree with edge set \( E = \{b_1a_1, a_1a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1\} \).

By a sequence of local switchings, \( (a_3, J = \{a_4\}) \), \( (a_4, J = \{a_2\}) \), \( (a_2, J = \{a_1, a_5\}) \), \( (a_1, J = \{a_2\}) \), \( (a_5, J = \{a_2\}) \), \( (a_2, J = \{a_1, a_5\}) \), \( (a_1, J = \{a_2\}) \), \( (a_5, J = \{a_2\}) \), \( (a_2, J = \{a_1, a_5\}) \), \( (a_1, J = \{a_2\}) \), \( (a_5, J = \{a_2\}) \), \( (a_2, J = \{a_1, a_5\}) \), \( (a_1, J = \{a_2\}) \), \( (a_5, J = \{a_2\}) \), \( (a_2, J = \{a_1, a_5\}) \), \( (a_1, J = \{a_2\}) \), \( (a_5, J = \{a_2\}) \), \( (a_2, J = \{a_1, a_5\}) \), \( (a_1, J = \{a_2\}) \), \( (a_5, J = \{a_2\}) \), \( (a_2, J = \{a_1, a_5\}) \), \( (a_1, J = \{a_2\}) \), \( (a_5, J = \{a_2\}) \), \( (a_2, J = \{a_1, a_5\}) \), \( (a_1, J = \{a_2\}) \), \( (a_5, J = \{a_2\}) \), \( (a_2, J = \{a_1, a_5\}) \), \( (a_1, J = \{a_2\}) \), \( (a_5, J = \{a_2\}) \), \( (a_2, J = \{a_1, a_5\}) \), from \( PH_2 \), we get a tree with edge set \( E = \{b_1a_1, a_1a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2, a_3b_3\} \).

By a sequence of local switchings, \( (a_3, J = \{a_4\}) \), \( (a_4, J = \{a_5\}) \), \( (a_2, J = \{a_1, a_5\}) \), from \( PH_3(1) \), we get a tree with edge set \( E = \{b_1a_1, a_1a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2, a_3b_3, a_4b_4\} \).

By sequence of local switchings, \( (a_4, J = \{a_5\}) \), \( (a_5, J = \{a_2\}) \), \( (a_2, J = \{a_1, a_5\}) \), from \( PH_5(1) \), we get a tree with edge set \( E = \{b_1a_1, a_1a_3, a_3a_4, a_4a_5, a_5a_1, a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5\} \).

We can not apply to this graph local switching at vertices \( a_1 \), or \( a_2 \), or \( a_3 \), or \( a_4 \), or \( a_5 \), or \( a_6 \), or \( b_3 \), or \( b_5 \) and can not transform it to a tree.

Define \( H_1, H_2(1), H_2(2), H_3(1), H_3(2), H_3(3) \) and \( H_4(1), H_4(2), H_4(3) \) as follows;

\( H_1 = (V, E), V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1\}, E^- = \{a_6a_1\} \);

\( H_2(1) = (V, E), V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_2b_2\}, E^- = \{a_6a_1\} \);
\[ H_2(2) = (V, E), V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_2b_2\}, E^- = \{a_6a_1\}; \]

\[ H_2(3) = (V, E), V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_2b_2, a_4b_2\}, E^- = \{a_6a_1\}; \]

\[ H_3(1) = (V, E), V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_2b_2, a_3b_3\}, E^- = \{a_6a_1\}; \]

\[ H_3(2) = (V, E), V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_2b_2, a_3b_3\}, E^- = \{a_6a_1\}; \]

\[ H_3(3) = (V, E), V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_2b_2, a_3b_3\}, E^- = \{a_6a_1\}; \]

\[ H_4(1) = (V, E), V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_1b_1, a_2b_2, a_3b_3\}, E^- = \{a_6a_1\}; \]

Now we obtain

**Proposition 8.** The graphs \(H_1, H_2(1), H_2(2), H_2(3), H_3(1), H_3(2), H_4(1)\) are transformed to trees, by a sequence of local switchings. The graphs \(H_3(3), H_4(2), H_4(3)\) cannot be transformed to trees by a sequence of local switchings.

**Proof.** By a sequence of local switchings, \((a_2, J = \{a_3\}), (a_3, J = \{a_4\}), (a_4, J = \{a_5\}), (a_5, J = \{a_6\})\), \((a_2, J = \{a_3\}), (a_3, J = \{a_4\}), (a_4, J = \{a_5\}), (a_5, J = \{a_6\})\), from \(H_1\), we get a tree with edge set \(E = \{b_1a_1, a_1a_5, a_5a_4, a_4a_3, a_3a_2\}\). By a sequence of local switchings, \((a_3, J = \{a_4\}), (a_4, J = \{a_5\}), (a_5, J = \{a_6\})\), \((a_4, J = \{a_5\}), (a_5, J = \{a_6\})\), from \(H_2(1)\), we get a tree with edge set \(E = \{b_1a_1, a_1a_5, a_5a_6, a_6a_2, a_5a_4, a_4a_3, a_3a_2\}\). By a sequence of local switchings, \((a_3, J = \{a_4\}), (a_4, J = \{a_5\}), (a_5, J = \{a_6\})\), \((a_4, J = \{a_5\}), (a_5, J = \{a_6\})\), from \(H_2(2)\), we get a tree with edge set \(E = \{b_1a_1, a_1a_5, a_5a_6, a_6a_2, a_5a_4, a_4a_3, a_3a_2\}\). By a sequence of local switchings, \((a_2, J = \{a_3\}), (a_3, J = \{a_4\}), (a_4, J = \{a_5\})\), \((a_2, J = \{a_3\}), (a_3, J = \{a_4\}), (a_4, J = \{a_5\})\), from \(H_3(1)\), we get a signed graph with edge sets \(E^+ = \{a_1a_3, a_3a_4, a_4a_5, a_5a_1, a_2a_3, a_4a_2, a_5a_6\}\), \(E^- = \{a_1a_5\}\) which is isomorphic to the signed graph \(QH_4\). Hence, \(H_2(3)\) is transformed to a tree, by a sequence of local switchings. By a sequence of local switchings, \((a_6, J = \{a_5\}), (a_5, J = \{a_4\}), (a_4, J = \{a_3\}), (a_5, J = \{a_6\}), (a_4, J = \{a_5\}), (a_3, J = \{a_1, b_2\}), (b_3, J = \{a_3\})\), from \(H_3(1)\), we get a tree with edge set \(E = \{b_1a_1, a_1a_3, a_3a_4, b_2a_2, b_2a_3, b_4a_4, a_4a_5, a_5a_6\}\). By a sequence of local switchings, \((a_6, J = \{a_5\}), (a_5, J = \{a_4\}), (a_4, J = \{a_3\}), (a_5, J = \{a_6\}), (a_4, J = \{a_5\}), (a_3, J = \{a_1, b_2\}), (b_3, J = \{a_4\})\), from \(H_2(2)\), we get a tree with edge set \(E = \{b_1a_1, a_1a_3, a_3a_4, b_2a_2, b_2a_3, b_4a_4, a_4a_5, a_5a_6\}\). By a sequence of local switchings, \((a_6, J = \{a_5\}), (a_5, J = \{a_4\}), (a_4, J = \{a_3\}), (a_5, J = \{a_6\}), (a_4, J = \{a_5\}), (a_3, J = \{a_1, b_2\}), (b_3, J = \{a_4\})\), from \(H_4(1)\), we get a tree with edge set \(E = \{b_1a_1, a_1a_3, a_3a_4, a_4a_5, a_5a_6, b_1a_4, b_4a_3, a_3a_2, a_2b_2\}\). By a sequence of local switchings, \((a_6, J = \{a_5\}), (a_5, J = \{a_4\}), (a_4, J = \{a_3\}), (a_5, J = \{a_6\}), (a_4, J = \{a_5\}), (a_3, J = \{a_1, b_2\}), (b_3, J = \{a_4\})\), from \(H_3(2)\), we get a tree with edge set \(E = \{b_1a_1, a_1a_3, a_3a_4, a_4a_5, a_5a_6, b_1a_4, b_4a_3, a_3a_2, a_2b_2\}\). By a sequence of local switchings, \((a_6, J = \{a_5\}), (a_5, J = \{a_4\}), (a_4, J = \{a_3\}), (a_5, J = \{a_6\}), (a_4, J = \{a_5\}), (a_3, J = \{a_1, b_2\}), (b_3, J = \{a_4\})\), from \(H_4(1)\), we get a tree with edge set \(E = \{b_1a_1, a_1a_3, a_3a_4, a_4a_5, a_5a_6, b_1a_4, b_4a_3, a_3a_2, a_2b_2\}\).
The graphs $H_3(3), H_4(2), H_4(3)$ can not be transformed to trees by any sequences of local switchings at all.

Define $C_{44}, C_{45}, C_{55}$ and $H_{55}$ as follows;

$C_{44} = (V,E), V = \{a_1, a_2, a_3, a_4, b_1\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_1b_1\}, E^- = \{a_3b_1\};$

$C_{45} = (V,E), V = \{a_1, a_2, a_3, a_4, a_5, b_1\}, E^+ = \{a_1a_2, a_3a_4, a_4a_5, a_5a_1, a_1b_1\}, a_3b_1}, E^- = \{a_2a_3\};$

$C_{55} = (V,E), V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_6a_1, a_1b_1, a_4b_1\}, E^- = \emptyset;$

$C_{56} = (V,E), V = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, b_1\}, E^+ = \{a_1a_2, a_2a_3, a_3a_4, a_5a_6, a_6a_7, a_1b_1, a_4b_1\}, E^- = \{a_2a_3\};$

Then, we get

**Proposition 9.** The graphs $C_{44}, C_{45}, C_{55}$ are transformed to trees, by a sequence of local switchings. The graph $C_{56}$ can not be transformed to a tree by a sequence of local switchings.

Proof. By a sequence of local switchings, $(a_1, J = \{b_1\}), (b_1, J = \{a_1, a_3\})$, from $C_{44}$, we get a tree with edge set $E = \{b_1a_1, b_1a_2, b_1a_3, b_1a_4\}$. By a sequence of local switchings, $(a_1, J = \{b_1\}), (b_1, J = \{a_2, a_5\}), (a_5, J = \{a_3\})$, from $C_{45}$, we get a tree with edge set $E = \{b_1a_1, b_1a_2, b_1a_3, a_3a_4\}$. By a sequence of local switchings, $(a_3, J = \{a_4\}), (a_5, J = \{a_6\}), (a_1, J = \{b_1\}), (b_1, J = \{a_2, a_6\}), (a_3, J = \{a_4\}), (a_5, J = \{a_3\}), (a_6, J = \{a_4\}), (a_2, J = \{a_4\}), (a_4, J = \{b_1\})$, from $C_{55}$, we get a tree with edge set $E = \{b_1a_1, b_1a_4, a_4a_3, a_4a_6, a_2a_3, a_6a_5\}$. The graph $C_{56}$ can not be transformed to a tree by any sequences of local switchings at all.

**References**


