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Axiomatic Method of Measure and Integration (I). Definition and Existence Theorem of the Jordan Measure

(Yoshifumi Ito "Differential and Integral Calculus II", Chapter 8)

By

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Abstract

In this paper, we define the Jordan measure on $\boldsymbol{R}^d$, $(d \geq 1)$ by prescribing the complete system of axioms. Then we prove the uniqueness and existence theorem of the Jordan measure. This is a new result.

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Introduction

This paper is the part I of the series of papers on the axiomatic method of measure and integration on the Euclidean space.

In this paper, we define the concept of the d-dimensional Jordan measure and its uniqueness and existence theorem. Here we assume $d \geq 1$. This is a new result.

A measure is one of the set functions.

In general, a function is defined by deciding its domain, its range and the rule of correspondence. Therefore, the set function as a measure is defined by prescribing the family of sets as its domain, its range and the additivity as the rule of correspondence.

Especially, the Jordan measure is a completely additive positive measure defined on a finitely additive family of the Jordan measurable sets and a invariant measure with respect to the group of congruent transformation. Here we say that the measure on the finitely additive family is completely additive if it is conditionally completely additive. Namely, the Jordan measure is completely additive in the sense that it is completely additive if the countable direct sum of the Jordan measurable sets is also a Jordan measurable set.

Especially, the Jordan measure is a finitely additive measure defined on the finitely additive family of all Jordan measurable sets.

In this paper, it is the new characterization that we define the Jordan measure by describing the complete system of axioms.

Further we prove the uniqueness and existence theorem of the Jordan measure. Thus, the definition of the Jordan measure and its uniqueness and existence theorem are the new results. We call this the axiomatic method of the measure and integration.

Until now, we construct the Jordan measure as one of set functions without defining the concept of the Jordan measure. Then there is one question that there is or not another measure than the well known Jordan measure. When we define the Jordan measure by giving the complete system of axioms, we can prove that there is the unique measure satisfying this system of axioms.

Thereby we know that there is no other Jordan measure than the measure constructed by Jordan himself. In this point, it is important that the theory of the Jordan measure is completed.

Here I show my heartfelt gratitude to my wife Mutuko for her help of typesetting this manuscript.

1 Definition of the Jordan measure

In this section, we define the concept of the Jordan measure. Here we assume $d \geq 1$.

We study the intervals and the blocks of intervals as the fundamental subsets in the d-dimensional Euclidean space $\boldsymbol{R}^d$.

We say that an interval I in $\boldsymbol{R}^d$ is the direct product set of the d intervals of the types

$$(a_i,\ b_i),\ [a_i,\ b_i),\ (a_i,\ b_i],\ [a_i,\ b_i],\ (i = 1,\ 2,\ \cdots,\ d)$$

in $\boldsymbol{R}$. Then we denote the interior of I by the symbol

$$I^\circ = (a_1,\ b_1) \times (a_2,\ b_2) \times \cdots \times (a_d,\ b_d).$$

Then I° is an open interval. Further we denote the closure of I by the symbol

$$\overline{I} = [a_1,\ b_1] \times [a_2,\ b_2] \times \cdots \times [a_d,\ b_d].$$

Then $\overline{I}$ is a closed interval. The empty set ϕ is considered as an interval.

If a finite number of intervals $I_1,\ I_2,\ \cdots,\ I_n$ are mutually disjoint, we say that the direct sum of them is a **block of intervals** and denote it as

$$E = \bigcup_{p=1}^{n} I_p = \sum_{p=1}^{n} I_p = I_1 + I_2 + \cdots + I_n.$$

We call this as the **decomposition** of the block of intervals.

Now we denote by $\mathcal{R}$ the family of all blocks of intervals in $\boldsymbol{R}^d$.

Then we have the following theorem 1.1.

Theorem 1.1 *Let $\mathcal{R}$ be the family of all blocks of intervals in $\boldsymbol{R}^d$. Then we have the following* (1)~(3):

(1) *$\phi \in \mathcal{R}$ holds.*

(2) *If $A \in \mathcal{R}$ holds, we have*

$$A^c = \{x \in \boldsymbol{R}^d;\ x \notin A\} \in \mathcal{R}.$$

(3) *If $A,\ B \in \mathcal{R}$ holds, we have $A \cup B \in \mathcal{R}$.*

Corollary 1.1 *Let $\mathcal{R}$ be the same as in Theorem* 1.1. *Then we have the following* (1), (2):

(1) *$\boldsymbol{R}^d \in \mathcal{R}$ holds.*

(2) *The set obtained by the finite times of operations such as the summation, the difference or the intersection of some sets in $\mathcal{R}$ belongs to $\mathcal{R}$. Here, the difference $A \backslash B$ of the sets A and B is defined by the relation*

$$A \backslash B = A \cap B^c = \{x \in \boldsymbol{R}^d;\ x \in A,\ x \notin B\}.$$

Then we define the Jordan measure space and the Jordan measure in the following definition 1.1.

Definition 1.1(the Jordan measure) We define that the triplet ($\boldsymbol{R}^d$, $\mathcal{B}$, μ) is the d-dimensional Jordan measure space if the family $\mathcal{B}$ of sets in the d-dimensional Euclidean space $\boldsymbol{R}^d$ and the set function μ on $\mathcal{B}$ satisfy the following axioms (I)~(III):

(I) $\mathcal{R} \subset \mathcal{B}$ holds.

(II) We have the following (i)∼(iv):

(i) For $A \in \mathcal{B}$, we have $0 \leq \mu(A) \leq \infty$.

(ii) If at most countable elements $A_1, \ A_2, \ \cdots, \ A_n, \ \cdots$ in $\mathcal{B}$ are mutually disjoint and their direct sum A satisfies the condition

$$A = \bigcup_{p=1}^{(\infty)} A_p = \sum_{p=1}^{(\infty)} A_p \in \mathcal{B},$$

then we have the equality

$$\mu(A) = \sum_{p=1}^{(\infty)} \mu(A_p).$$

(iii) For $I_0 = [0, \ 1]^d$, we have $\mu(I_0) = 1$. Here $[0, \ 1]^d$ denotes the direct product set of the d closed intervals $[0, \ 1]$.

(iv) When we define $A + x = \{y + x; \ y \in A\}$ is the translated set of a set $A \in \mathcal{B}$ for a vector $x \in \boldsymbol{R}^d$, then we have $A + x \in \mathcal{B}$ and $\mu(A + x) = \mu(A)$ holds.

(III) $A \in \mathcal{B}$ if and only if, for any bounded set $E \in \mathcal{B}$, we have the equality

$$\mu^*(A \cap E) = \mu_*(A \cap E).$$

Then we have the equality

$$\mu(A) = \sup\{\mu^*(A \cap E); \ E \in \mathcal{R} \text{ is bounded}\}.$$

Here μ^* and μ_* denotes the outer measure and the inner measure respectively which are defined by the measure μ on $\mathcal{R}$ obtained by the restriction of μ on $\mathcal{B}$. Namely, $\mu^*(A \cap E)$ and $\mu_*(A \cap E)$ are defined by the formulas

$$\mu^*(A \cap E) = \inf\{\mu(B); \ B \supset A \cap E, \ B \in \mathcal{R}\},$$

$$\mu_*(A \cap E) = \sup\{\mu(B); \ A \cap E \supset B, \ B \in \mathcal{R}\}$$

respectively.

Then we say that an element in $\mathcal{B}$ is a **Jordan measurable set** and μ is the **d-dimensional Jordan measure**.

For simplicity, we call the d-dimensional Jordan measure space and the d-dimensional Jordan measure as the **Jordan measure space** and the **Jordan measure** respectively.

Further, we call a Jordan measurable set as a measurable set.

The symbol

$$\bigcup_{p=1}^{(\infty)} A_p = \sum_{p=1}^{(\infty)} A_p$$

used in the Axiom (II), (ii) denotes the finite or countable sum of the sets A_p, and the symbol

$$\sum_{p=1}^{(\infty)} \mu(A_p)$$

denotes the finite or countable sum of the measures $\mu(A_p)$. This condition of Axiom (II), (ii) means that the d-dimensional Jordan measure is a conditionally completely additive measure.

The d-dimensional Jordan measure considered here is a finitely additive measure without any condition. But it becomes a conditionally completely additive measure by virtue of, what is called, the condition of continuity.

By virtue of this condition, we extend the d-dimensional Jordan measure to the d-dimensional Lebesgue measure. Therefore it is possible to complete the d-dimensional Jordan measure. In this paper, in the construction of the theory of the Riemann integral, we do not use this fact afterward.

In the following, we prove the uniqueness and existence theorem of the d-dimensional Jordan measure in Definition 1.1.

In order to do this, we determine concretely the family $\mathcal{B}$ of the Jordan measurable sets and the Jordan measure μ on $\boldsymbol{R}^d$.

At first, we study the characterization of the Jordan measurable sets. As the necessary condition that the set function μ in Definition 1.1 is the d-dimensional Jordan measure, we determine concretely the value $\mu(A)$ of each element A in $\mathcal{B}$.

Thereby, we determine the family $\mathcal{B}$ of the Jordan measurable sets. Then, we prove that the value of the set function μ determined like that satisfies the condition of the definition of the d-dimensional measure in Definition 1.1. Thus we prove the uniqueness and existence of the d-dimensional Jordan measure.

Here we continue to characterize the d-dimensional Jordan measure derived from Definition 1.1.

2 Definition of the Jordan measure of the blocks of intervals

We say that the function μ on $\mathcal{R}$ which defines a real number $\mu(A)$ for a set A in $\mathcal{R}$ is a set function on $\mathcal{R}$.

By restricting the Jordan measure in Definition 1.1 to the family $\mathcal{R}$ of all blocks of intervals, we have the following concept of the Jordan measure of a block of intervals.

Definition 2.1 Let $\mathcal{R}$ be the same as in Definition 1.1. Then, a set function μ on $\mathcal{R}$ is defined to be the **Jordan measure** of the blocks of intervals in $\boldsymbol{R}^d$ if the following (i)~(iv) hold:

(i) If $A \in \mathcal{R}$ holds, we have $0 \leq \mu(A) \leq \infty$.

(ii) If at most countable elements $A_1,\ A_2,\ \cdots, A_n,\ \cdots$ in $\mathcal{R}$ are mutually disjoint and the condition

$$A = \bigcup_{p=1}^{(\infty)} A_p = \sum_{p=1}^{(\infty)} A_p \in \mathcal{R}$$

is satisfied, we have the equality

$$\mu(A) = \sum_{p=1}^{(\infty)} \mu(A_p).$$

(iii) If $I_0 = [0,\ 1]^d$ holds, we have $\mu(I_0) = 1$.

(iv) If $E + x$ is the set of the translation of a set $E \in \mathcal{R}$ for a vector $x \in \boldsymbol{R}^d$, we have $E + x \in \mathcal{R}$ and the equality $\mu(E + x) = \mu(E)$.

Then, the value $\mu(E)$ of μ at $E \in \mathcal{R}$ is said to be the **Jordan measure** of a block of intervals E.

Remark 2.1 When we define the unit set to be $I_0 = [0,\ 1]^d$ in Definition 2.1, (iii), it means that we consider one orthogonal coordinate system in the d-dimensional Euclidean space.

Corollary 2.1 *For the Jordan measure μ of the blocks of intervals in $\boldsymbol{R}^d$, we have the following* (1)~(4):

(1) *If the elements $A_1,\ A_2, \cdots A_n$ in $\mathcal{R}$ are mutually disjoint, the condition*

$$A = \bigcup_{p=1}^{n} A_p = \sum_{p=1}^{n} A_p \in \mathcal{R}$$

is satisfied and we have the equality

$$\mu(A) = \sum_{p=1}^{n} \mu(A_p).$$

(2) *If $A \supset B$ holds for $A,\ B \in \mathcal{R}$, we have the inequality*

$$\mu(A) \geq \mu(B).$$

Especially, if $\mu(A) < \infty$ holds, we have the equality

$$\mu(A \backslash B) = \mu(A) - \mu(B).$$

Especially we have

$$\mu(\phi) = 0.$$

(3) *If the condition*

$$A = \bigcup_{p=1}^{(\infty)} A_p \in \mathcal{R}$$

is satisfied for at most countable elements $A_1,\ A_2,\ \cdots$ in $\mathcal{R}$, we have the inequality

$$\mu(A) \leq \sum_{p=1}^{(\infty)} \mu(A_p).$$

(4) *If at most countable intervals $I_1,\ I_2,\ \cdots,\ I_n,\ \cdots$ are mutually disjoint, and the set*

$$I = \bigcup_{p=1}^{(\infty)} I_p = \sum_{p=1}^{(\infty)} I_p$$

is also an interval, we have the equality

$$\mu(I) = \sum_{p=1}^{(\infty)} \mu(I_p).$$

3 Proof of the existence theorem of the Jordan measure of the blocks of intervals

At first, we concretely determine the Jordan measure μ of the blocks of intervals. Thus we prove the existence theorem of the Jordan measure of the blocks of intervals.

Here, by the similar way to the existence proof of the Jordan measure of the blocks of intervals in $\boldsymbol{R}$, we prove the existence of the Jordan measure of the blocks of intervals in $\boldsymbol{R}^d$.

Therefore, we have the following propositions.

Proposition 3.1 *The Jordan measure of the closed intervals* $[0,\ 1/n]^d$ *or the intervals obtained by removing its parts or the whole of the boundary is equal to* $1/n^d$. *Here we assume* $n \geq 1$.

Proposition 3.2 *The Jordan measure of a closed interval* $[0,\ a_1]\times[0,\ a_2]\times \cdots \times [0,\ a_d]$ *or the intervals obtained by removing its parts or the whole of the boundary is equal to* $a_1 a_2 \cdots a_d$. *Here we assume* $a_1,\ a_2,\ \cdots,\ a_d \geq 0$.

Remark 3.1 When we calculate the limit for the d-dimensional measure in the proof of Proposition 3.2, we note that we use the conditionally complete additivity.

Corollary 3.1 *If the interval* I *is the closed interval* $[a_1,\ b_1] \times [a_2,\ b_2] \times \cdots \times [a_d,\ b_d]$ *or the interval obtained by removing its parts or the whole of the boundary, the Jordan measure of* I *is equal to*

$$\mu(I) = (b_1 - a_1)(b_2 - a_2) \cdots (b_d - a_d).$$

Here we assume that $a_i,\ b_i,\ (i = 1,\ 2,\ \cdots,\ d)$ *are real numbers and* $a_i \leq b_i,\ (i = 1,\ 2,\ \cdots,\ d)$ *hold.*

Corollary 3.2 *The Jordan measure of the unbounded interval* I *is equal to the following* (1) *or* (2):

(1) *When* I *is not included in the hyperplane which is parallel to some coordinate axis, we have*

$$\mu(I) = \infty.$$

(2) *When* I *is included in the hyperplane which is parallel to some coordinate axis, we have*

$$\mu(I) = 0.$$

Proposition 3.3 *If a block of intervals* E *in* $\mathcal{R}$ *is equal to the direct sum*

$$E = I_1 + I_2 + \cdots + I_n \tag{3.1}$$

of mutually disjoint intervals $I_1,\ I_2,\ \cdots,\ I_n$, *the Jordan measure* $\mu(E)$ *of* E *is equal to*

$$\mu(E) = \mu(I_1) + \mu(I_2) + \cdots + \mu(I_n). \tag{3.2}$$

Then the formula (3.2) *does not depend on the way of the decomposition of* E *by the mutually disjoint intervals.*

Remark 3.2 If a block of intervals E is unbounded, at least one interval is unbounded in the direct sum decomposition (3.1).

Then we note that it happens to be $\mu(E) = \infty$ by virtue of Corollary 3.2.

Conversely, we have the following theorem concerning the existence of the Jordan measure of the blocks of intervals.

Theorem 3.1 *We define the set function μ on $\mathcal{R}$ in the following*:

(i) *For a bounded closed interval $[a_1,\ b_1]\times[a_2,\ b_2]\times\cdots\times[a_d,\ b_d]$ or a interval I obtained by removing the part or the whole of its boundary, we define $\mu(I) = (b_1 - a_1)(b_2 - a_2)\cdots(b_d - a_d)$. Here we assume that $a_i,\ b_i,\ (i = 1,\ 2,\ \cdots,\ d)$ are some real numbers such as $a_i \leq b_i,\ (i = 1,\ 2,\ \cdots,\ d)$ hold.*

(ii) *The Jordan measure of a unbounded interval I is defined by the following*:

(1°) *When I is not included in a certain hyperplane which is parallel to a certain coordinate axis, we define*

$$\mu(I) = \infty.$$

(2°) *When I is included in a certain hyperplane which is parallel to a certain coordinate axis, we define*

$$\mu(I) = 0.$$

(iii) *If we decompose a block of intervals A as the direct sum of intervals $I_1,\ I_2,\ \cdots,\ I_n$ and denote it as*

$$A = I_1 + I_2 + \cdots + I_n,$$

we define

$$\mu(A) = \mu(I_1) + \mu(I_2) + \cdots + \mu(I_n).$$

(iv) *If at most countable intervals $I_1,\ I_2,\ I_3,\ \cdots$ are mutually disjoint and*

$$I = \bigcup_{p=1}^{(\infty)} I_p = \sum_{p=1}^{(\infty)} I_p$$

is also an interval, we assume that the relation

$$\mu(I) = \sum_{p=1}^{(\infty)} \mu(I_p)$$

holds.

Then $\mu(A)$ is the Jordan measure of the blocks of intervals.

4 Proof of the existence theorem of the Jordan measure

In this section, we prove the existence theorem of the Jordan measure.

Here we concretely determine the d-dimensional Jordan measure μ. Thereby we prove the existence theorem of the d-dimensional Jordan measure.

For that purpose, we prepare the following.

Definition 4.1 For an arbitrary subset B of $\boldsymbol{R}^d$, we define that

$$\mu^*(B) = \inf\{\mu(A);\ B \subset A,\ A \in \mathcal{R}\},$$

$$\mu_*(B) = \sup\{\mu(A);\ A \subset B,\ A \in \mathcal{R}\}.$$

are the **outer measure** and the **inner measure** of B respectively.

Corollary 4.1 *For $A \in \mathcal{R}$, we have the following equality*

$$\mu^*(A) = \mu_*(A) = \mu(A).$$

Here the third side denote the Jordan measure of a block of intervals in the sense of Theorem 3.1.

By virtue of the definitions of the outer measure and the inner measure, we have the following three propositions immediately.

In the following, let $B,\ B_1,\ B_2$ be some subsets of $\boldsymbol{R}^d$.

Proposition 4.1 *We have the inequality*

$$0 \leq \mu_*(A) \leq \mu^*(B) \leq \infty.$$

Especially, we have the equality

$$\mu^*(\phi) = \mu_*(\phi) = 0.$$

Proposition 4.2 *If $B_1 \subset B_2$ holds, we have the following* (1) *and* (2):

(1) $\mu^*(B_1) \leq \mu^*(B_2)$. (2) $\mu_*(B_1) \leq \mu_*(B_2)$.

Proposition 4.3 *We have the following inequality*

$$\mu^*(B_1 \cup B_2) \leq \mu^*(B_1) + \mu^*(B_2).$$

Proposition 4.4 *If we put*

$$A = \bigcup_{p=1}^{(\infty)} A_p$$

for at most countable subsets $A_1,\ A_2,\ A_3,\ \cdots$ *of* $\boldsymbol{R}^d$, *we have the inequality*

$$\mu^*(A) \leq \sum_{p=1}^{(\infty)} \mu^*(A_p).$$

Proposition 4.5 *If at most countable subsets* $A_1,\ A_2,\ A_3,\ \cdots$ *of* $\boldsymbol{R}^d$ *are mutually disjoint and we put*

$$A = \sum_{p=1}^{(\infty)} A_p,$$

we have the inequality

$$\mu_*(A) \geq \sum_{p=1}^{(\infty)} \mu_*(A_p).$$

Proposition 4.6 *If* B *is an arbitrary subset* B *in* $\boldsymbol{R}^d$, *we have the equality*

$$\mu_*(B \cap E) = \mu(E) - \mu^*(B^c \cap E)$$

for an arbitrary bounded set $E \in \mathcal{R}$. *Here* μ *denote the Jordan measure of the blocks of intervals defined in Theorem* 3.1.

Proposition 4.7 *Let* B *be an arbitrary subset of* $\boldsymbol{R}^d$. *If* $E_1,\ E_2,\ \cdots$ *are some sequence of bounded blocks of intervals of* $\boldsymbol{R}^d$ *which satisfy the conditions*

$$E_1 \subset E_2 \subset \cdots,\ \bigcup_{n=1}^{\infty} E_n = \boldsymbol{R}^d.$$

Then we have the equalities

$$\mu^*(B) = \lim_{n\to\infty} \mu^*(B \cap E_n), \tag{4.5}$$

$$\mu_*(B) = \lim_{n\to\infty} \mu_*(B \cap E_n). \tag{4.6}$$

Definition 4.2 We say that an arbitrary subset B of $\boldsymbol{R}^d$ is **measurable** if we have the equality $\mu^*(B \cap E) = \mu_*(B \cap E)$ for an arbitrary bounded set $E \in \mathcal{R}$. Then we say that

$$\mu(B) = \sup\{\mu^*(B \cap E);\ E \text{ is an arbitrary bounded block of intervals}\}$$

is the **Jordan measure** of B.

Remark 4.1 The measurability of a subset B of $\boldsymbol{R}^d$ means that, for any bounded part of B, the outer measure $\mu^*(B \cap E)$ which is the approximation of the measures of bounded blocks of intervals from outer side and the inner measure $\mu_*(B \cap E)$ which is the approximation of the measures of bounded blocks of intervals from inner side are both identical.

Corollary 4.2 *If B is an arbitrary measurable set of $\boldsymbol{R}^d$, we have the equalities*

$$\mu^*(B) = \mu_*(B) = \mu(B).$$

By virtue of Corollary 4.1, the concept of the Jordan measure of the measurable sets are identical to the concept of the Jordan measure of the blocks of intervals for any blocks of intervals.

Now we show some examples of non measurable sets.

Example 4.1 The set

$$B = [0,\ 1]^d \cup \{\text{the rational points in } [0,\ 1]^{d-1} \times [1,\ 2]\}$$

is not measurable. Here a rational point means a point with the coordinates which are the rational numbers.

Proof Because we have the relation

$$\mu^*(B) = 2 \neq \mu_*(B) = 1$$

for the set B, B is not measurable. //

Example 4.2 The set

$$B = \{\text{the rational points in } [0,\ 1]^d\} \cup ([0,\ 1]^{d-1} \times [1,\ \infty])$$

is not measurable.

Proof We consider one bounded part

$$B \cap ([0,\ 1]^{d-1} \times [0,\ 2]) = \{\text{the rational points in } [0,\ 1]^d\}$$

$$\cup([0,\ 1]^{d-1} \times [1,\ 2]).$$

By the similar way to Example 4.1, we have the relation

$$\mu^*(B \cap ([0,\ 1]^{d-1} \times [0,\ 2])) = 2 \neq \mu_*(B \cap ([0,\ 1]^{d-1} \times [0,\ 2])) = 1.$$

Thus B does not satisfy the condition of Definition 4.2. Therefore B is not measurable. //

Remark 4.2 For the set B in Example 4.2, we have the equalities

$$\mu^*(B) = \mu_*(B) = \infty.$$

Although the outer measure and the inner measure of B are identical, B is an example which is not said to be measurable only by this condition.

This is the reason why the condition that the outer measure and the inner measure of every bounded part of B is the condition of measurability.

In the following, we show that the set function μ defined in Definition 4.2 satisfies the condition of the Jordan measure in Definition 1.1.

Theorem 4.1 *Let B be an arbitrary subset of $\boldsymbol{R}^d$. Then B is measurable if and only if we have the equality*

$$\mu^*(B \cap E) + \mu^*(B^c \cap E) = \mu(E)$$

for an arbitrary $E \in \mathcal{R}$.

Theorem 4.2 *Let B be an arbitrary subset of $\boldsymbol{R}^d$. Then B is measurable if and only if we have the equality*

$$\mu^*(B \cap A) + \mu^*(B^c \cap A) = \mu^*(A)$$

for an arbitrary subset A of $\boldsymbol{R}^d$.

Theorem 4.3 *Let B be an arbitrary subset of $\boldsymbol{R}^d$. Then B is measurable if and only if we have the equality*

$$\mu^*(A_1 + A_2) = \mu^*(A_1) + \mu^*(A_2)$$

for arbitrary two subsets $A_1,\ A_2$ such that $A_1 \subset B$ and $A_2 \subset B^c$ hold.

Theorem 4.4 *Let $\dot{B}$ be the boundary of an arbitrary bounded subset B of $\boldsymbol{R}^d$. Then we have the equality*

$$\mu^*(\dot{B}) = \mu^*(B) - \mu_*(B).$$

Then we have the following.

Theorem 4.5 *Let B be an arbitrary subset of $\boldsymbol{R}^d$. Then B is measurable if and only if we have the equality*

$$\mu^*(\dot{B}) = 0$$

for the boundary $\dot{B}$ of B.

Because an arbitrary subset in $\boldsymbol{R}^d$ with the outer measure 0 has necessarily the Jordan measure 0, we have the following.

Theorem 4.6 *An arbitrary subset of $\boldsymbol{R}^d$ is measurable if and only if its boundary has the Jordan measure* 0.

Theorem 4.7 *Let $\mathcal{B}$ be the family of all measurable subsets of $\boldsymbol{R}^d$. Then $\mathcal{B}$ has the following* (1)$\sim$(3):

(1) *$\mathcal{R} \subset \mathcal{B}$ holds. Especially we have $\phi \in \mathcal{B}$.*

(2) *If $A \in \mathcal{B}$ holds, we have $A^c \in \mathcal{B}$.*

(3) *If $A,\ B \in \mathcal{B}$ holds, we have $A \cup B \in \mathcal{B}$.*

Corollary 4.3 *Let $\mathcal{B}$ be the same as in Theorem* 4.7. *Then we have the following* (1), (2):

(1) *$\boldsymbol{R}^d \in \mathcal{B}$ holds.*

(2) *The sets obtained by the finite times of operations such as the summation, the difference and the intersection of sets in $\mathcal{B}$ belong to $\mathcal{B}$.*

Proposition 4.8 *If a bounded subset B in $\boldsymbol{R}^d$ has the boundary which is composed of the finite number of C^1-hypersurfaces, then B is measurable.*

Theorem 4.8 *The property that a certain subset in $\boldsymbol{R}^d$ is measurable does not depend on the choice of an orthogonal coordinate system. Further, the Jordan measure of a measurable set does not depend on the choice of an orthogonal coordinate system.*

Theorem 4.9 *If two measurable sets in $\boldsymbol{R}^d$ are congruent, their Jordan measures are equal.*

Theorem 4.10 *If $A,\ B \in \mathcal{B}$ and $A \cap B = \phi$ hold, we have the equality*

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Corollary 4.4 *If at most countable sets $A_1,\ A_2,\ A_3,\ \cdots$ in $\mathcal{B}$ are mutually disjoint and they satisfy the condition*

$$A = \bigcup_{p=1}^{(\infty)} A_p = \sum_{p=1}^{(\infty)} A_p \in \mathcal{B},$$

we have the equality

$$\mu(A) = \sum_{p=1}^{(\infty)} \mu(A_p).$$

By virtue of the considerations in the above, we have the following existence theorem of the Jordan measure.

Theorem 4.11(Uniqueness and existence theorem) *There exists only one Jordan measure space $(\boldsymbol{R}^d,\ \mathcal{B},\ \mu)$ on $\boldsymbol{R}^d$. Here $\mathcal{B}$ denotes the family of all measurable sets on $\boldsymbol{R}^d$ and the set function μ defined on $\mathcal{B}$ denotes the d-dimensional Jordan measure.*

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