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Axiomatic Method of Measure and Integration (II). Definition of the Riemann Integral and its Fundamental Properties

(Yoshifumi Ito "Differential and Integral Calculus II", Chapter 9)

By

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Abstract

In this paper, we define the Riemann integral of the Jordan measurable function on $\boldsymbol{R}^d, (d \geq 1)$.

Then we study the method of calculation of the Riemann integral. Further we clarify the convergence properties of the Riemann integral completely. These facts are the new results.

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Introduction

This paper is the part II of the series of papers for the study of the axiomatic method of measure and integration on the Euclidean space. As for the details, we refer to [1], [3]∼[6], [8]∼[10].

In this paper, we define the Riemann integral of Jordan measurable functions on the Jordan measure space $(E,\ \mathcal{B}_E,\ \mu)$.

Let E be a measurable set in $\boldsymbol{R}^d$. Then we define that a function $f(x) = f(x_1,\ x_2,\ \cdots,\ x_d)$ defined on E is Jordan measurable if it is the Moore-Smith limit in the sense of uniform convergence of a direct family $\{f_\Delta(x)\}$ of simple functions in the wider sense on E. Then we define that the Riemann integral of a measurable function $f(x)$ on E is the Moore-Smith limit

$$\int_E f(x)dx = \lim_{\Delta} \int_E f_\Delta(x)dx$$

of the direct set of the Riemann integrals of simple functions.

The Riemann integral on E either converges or diverges. The convergence of the Riemann integral is either the absolute convergence or the conditional convergence. When the Riemann integral converges conditionally, we have said until now that this is the improper Riemann integral.

As for the fundamental properties of the Riemann integral, we prove these for simple functions first. For general measurable functions, we prove these properties considering them as the limit of the direct family of simple functions.

The outline is that the Jordan measurable functions and the Lebesgue measurable functions are the limits of simple functions. As for their difference points, other than the definitions of measurable sets, it is essential that the topologies defining the limits of simple functions are different.

The outline is that a Jordan measurable function is the limit of simple functions in the sense of the uniform convergence in the wider sense and, on the other hand, a Lebesgue measurable function is the limit of simple functions in the sense of pointwise convergence. By considering in such a framework, it is the point of improvement that we can study the Riemann integral and the Lebesgue integral by the similar method at many points.

Here I show my heartfelt gratitude to my wife Mutuko for her help of typesetting this manuscript.

1 Measurable functions

In this section, we define the concept of the measurable functions on $\boldsymbol{R}^d$ and study their fundamental properties. Here we assume $d \geq 1$.

In the next section, we define the concept of the Riemann integral of a measurable function on $\boldsymbol{R}^d$. In the definition of a measurable function and the Riemann integral, we have to consider the limit in the sense of Moore-Smith limit. We assume that the d-dimensional Jordan measure space $(\boldsymbol{R}^d,\ \mathcal{B},\ \mu)$ is defined on $\boldsymbol{R}^d$. We denote a point in $\boldsymbol{R}^d$ as $x = (x_1,\ x_2,\ \cdots,\ x_d)$.

Then we denote a function on $\boldsymbol{R}^d$ as $f(x) = f(x_1, x_2, \cdots, x_d)$. We assume that the value of $f(x)$ is a real number or $\pm\infty$. Then we say that such a function is an **extended real valued function**.

Now we assume that a set E in $\boldsymbol{R}^d$ is a **measurable set**. Then we say that E is measurable for simplicity.

Here we divide a set E into the countable measurable subsets $E_1, E_2, E_3, \cdots$. Namely, we assume that $E_1, E_2, E_3, \cdots$ are mutually disjoint and their sum is equal to E. We denote this as

$$(\Delta): \ E = E_1 + E_2 + E_3 + \cdots .$$

In the sequel, we consider only the division of E into the countable subsets using the measurable subsets. We call this the division of E for simplicity.

For two divisions Δ and Δ' of E, we say that Δ' is a finer division than Δ if every small subset of Δ' is included in a certain small subset of Δ. We obtain the finer division of Δ if we divide a certain small subset of the division Δ into two measurable subsets. An arbitrary finer division of Δ is obtained by the countable iteration of the division of a small subset of Δ into two subsets.

For two divisions Δ and Δ' of E, we denote $\Delta \leq \Delta'$ if Δ' is a finer division of Δ.

Now we denote the family of all divisions of E as $\boldsymbol{\Delta} = \boldsymbol{\Delta}(E)$. This relation $\leq$ is an order of $\boldsymbol{\Delta}$.

This is a direct set with respect to the order defined by the finer division.

Namely, this means that we have the following conditions (1)$\sim$ (4):

Here, we assume $\Delta, \Delta', \Delta'' \in \boldsymbol{\Delta}$.

(1) For all $\Delta \in \boldsymbol{\Delta}$, we have $\Delta \leq \Delta$.

(2) If $\Delta \leq \Delta', \ \Delta' \leq \Delta$ hold, we have $\Delta = \Delta'$.

(3) If $\Delta \leq \Delta', \ \Delta' \leq \Delta''$ hold, we have $\Delta \leq \Delta''$.

(4) For two arbitrary $\Delta, \Delta' \in \boldsymbol{\Delta}$, there exists a certain $\Delta'' \in \boldsymbol{\Delta}$ such that we have $\Delta \leq \Delta'', \ \Delta' \leq \Delta''$.

These fact can be proved as follows.

Because it is evident that (1)$\sim$(3) hold, we prove that (4) holds. Now we assume that we have the two divisions $\Delta, \Delta' \in \boldsymbol{\Delta}$ of E such as

$$(\Delta): \ E = E_1 + E_2 + E_3 + \cdots ,$$

$$(\Delta'): \ E = F_1 + F_2 + F_3 + \cdots .$$

Then, if we define the division Δ'' as follows:

$$(\Delta''): \ E = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} E_i \cap F_j ,$$

it is evident that we have $\Delta'' \in \boldsymbol{\Delta}$ and $\Delta \leq \Delta''$, $\Delta' \leq \Delta''$ hold.

Then $(E,\ \mathcal{B}_E,\ \mu)$ is the Jordan measure space. Here $\mathcal{B}_E$ is the family of all Jordan measurable sets included in E and μ is the Jordan measure restricted on $\mathcal{B}_E$.

In general, when we have the Jordan measure space $(E,\ \mathcal{B}_E,\ \mu)$, we consider the measurable functions defined on E.

Here we define the simple functions.

Definition 1.1 We say that a function $f(x) = f(x_1,\ x_2,\ \cdots,\ x_d)$ defined on a measurable set E is a **simple function** if $f(x)$ is defined by the relation

$$f(x) = \sum_{j=1}^{\infty} \alpha_j \chi_{E_j}(x) \tag{1.1}$$

for a division Δ of E:

$$(\Delta):\ E = E_1 + E_2 + E_3 + \cdots . \tag{1.2}$$

Here α_j is equal to a real number, $(1 \leq j < \infty)$ and they have not to be mutually different. $\chi_{E_j}(x)$ is the defining function of the set E_j, $(1 \leq j < \infty)$. Then we denote this simple function $f(x)$ as $f_\Delta(x)$.

Remark 1.1 A function $f(x)$ is defined by prescribing its domain, its range and the rule of correspondence defining it. Then, the range of the simple function $f(x)$ is determined as an at most countable set of real numbers. But, as for the representation of the simple function such as the formula (1.1), there are infinitely many types of representation corresponding to the many types of divisions of E such as the formula (1.2).

Then we define a measurable function as follows. Here $\overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{\pm\infty\}$ denotes the extended space of real numbers.

Definition 1.2 Let E be a measurable set in $\boldsymbol{R}^d$. Then we define that a extended real valued function $f(x)$ defined on E is a **measurable function** if it satisfies the following conditions (i) and (ii):

(i) When we put $E(\infty) = \{x \in E;\ |f(x)| = \infty\}$, we have $E(\infty) \in \mathcal{B}$ and $\mu(E(\infty)) = 0$.

(ii) There exists a direct family $\{f_\Delta(x);\ \Delta \in \boldsymbol{\Delta}\}$ of simple functions such that we have the Moore-Smith limit

$$\lim_{\Delta} f_\Delta(x) = f(x) \tag{1.3}$$

which converges uniformly on $E \backslash E(\infty)$ in the wider sense.

Definition 1.2, (ii) is equivalent to the following condition (iii):

(iii) For an arbitrary measurable bounded closed set K in $E\backslash E(\infty)$ and an arbitrary positive number $\varepsilon > 0$, there exists some $\Delta_0 \in \boldsymbol{\Delta}$ such that we have the inequality

$$|f_\Delta(x) - f(x)| < \varepsilon, \ (x \in K) \tag{1.4}$$

for any $\Delta \in \boldsymbol{\Delta}$ such as $\Delta_0 \leq \Delta$.

For the simplicity of expression, we say that a measurable function $f(x)$ is **measurable**.

Remark 1.2 The Moore-Smith limit in the formula (1.3) means that the uniform convergence of $f_\Delta(x)$ to $f(x)$ in the wider sense when we continue to divide $E\backslash E(\infty)$ into the countable subsets as being finer, more finer and infinitely finer.

Remark 1.3 When the division of a measurable set E is given by the formula (1.2), we denote the diameter of E_i by the symbol

$$\delta_i = \sup\{d(p, \ q); \ p, \ q \in E_i\}, \ (1 \leq i < \infty).$$

Then, if we put

$$\delta = \delta(\Delta) = \sup_{1 \leq i < \infty} \delta_i,$$

we have the equality

$$\lim_{\Delta \in \boldsymbol{\Delta}} \delta(\Delta) = 0$$

in the sense of the Moore-Smith limit.

Remark 1.4 Now we assume that $\boldsymbol{\Delta}_0$ is a subfamily of $\boldsymbol{\Delta}$ and

$$\lim_{\Delta \in \boldsymbol{\Delta}_0} \delta(\Delta) = 0$$

in the sense of the Moore-Smith limit.

But, we have not to assume that $\boldsymbol{\Delta}_0$ and $\boldsymbol{\Delta}$ are cofinal. In general, the fact that we have

$$\lim_{\delta \to 0} \delta(\Delta) = 0$$

is not a topological property for the family $\boldsymbol{\Delta}$ of divisions of E.

In general, the limit must be a topological property. Therefore it is the reason why we must use the Moore-Smith limit in order to define the Riemann integral.

Example 1.1 Let a set E be a measurable set in $\boldsymbol{R}^d$. Then a simple function and a continuous function $f(x)$ defined on E are measurable.

Theorem 1.1 *Let a set E be a measurable set in $\boldsymbol{R}^d$. Assume that two functions f and g are measurable on E. Then the following functions* (1)∼(10) *defined on E are also measurable*:

(1) $f+g$. (2) $f-g$. (3) fg.

(4) f/g. *Here, for every measurable bounded closed set K in E, there exists some constant $\delta > 0$ such that we have $|g| \geq \delta$, $(x \in K)$.*

(5) αf. *Here α is a real constant.*

(6) $|f|$. (7) $\sup(f,\ g)$. (8) $\inf(f,\ g)$.

(9) $f^+ = \sup(f,\ 0)$. (10) $f^- = -\inf(f,\ 0)$.

Theorem 1.2 *Assume that a set E is a measurable set in $\boldsymbol{R}^d$. Assume that there exists a sequence $\{f_n\}$ of measurable functions on E and there exists a certain set E_0 of measure 0 such that the sequence $\{f_n\}$ converges to f uniformly on $E \backslash E_0$ in the wider sense. Then the function f is measurable on E.*

Example 1.2 Let $f(x)$ be a function defined on the closed interval $[0,\ 1]^d$ which is equal to 1 for a rational point x and equal to 0 for the other point. Then $f(x)$ is not measurable. Here we say that x is a rational point if all the coordinates $x_1,\ x_2,\ \cdots,\ x_d$ are rational numbers.

2 Definition of the Riemann integral

In this section, we define the Riemann integral of a measurable function of d variables.

We assume that the d-dimensional Jordan measure space $(\boldsymbol{R}^d,\ \mathcal{B},\ \mu)$ is defined on $\boldsymbol{R}^d$. In general, we assume that the domain of integration is a d-dimensional measurable set which is not necessarily an interval. Further a subset E of $\boldsymbol{R}^d$ is a general measurable set which is not necessarily bounded.

Then we define the Riemann integral of a extended real valued measurable function $f(x)$ defined on E.

We define the Riemann integral in the following two cases.

(1) The case where $f(x)$ is a simple function

In this case, $f(x)$ is defined in the formula

$$f(x) = \sum_{j=1}^{\infty} \alpha_j \chi_{E_j}(x) \tag{2.1}$$

for a division of E:

$$(\Delta):\ E = E_1 + E_2 + E_3 + \cdots. \tag{2.2}$$

Here we have $\alpha_j \in \boldsymbol{R},\ (1 \leq j < \infty)$.

Then we define the integral of $f(x)$ on E by the formula

$$\int_E f(x)dx = \sum_{j=1}^{\infty} \alpha_j \mu(E_j). \tag{2.3}$$

Here we assume that the series in the right hand side converges absolutely.

The value of the right hand side of the formula (2.3) is determined uniquely independent to the expression of the simple function $f(x)$ as in the formula (2.1). Therefore, the definition of the Riemann integral by using the formula (2.3) is meaningful.

The symbol of the integral in the left hand side of the formula (2.3) is the abbreviation of the following symbol of the Riemann integral

$$\int\int\cdots\int_E f(x_1,\ x_2,\ \cdots,\ x_d)dx_1dx_2\cdots dx_d.$$

In the sequel, we use this simplified symbol without a special mention.

(2) The case where $f(x)$ is a general measurable function

In this case, we assume that a function $f(x)$ is a general measurable function defined on E. Then we have a direct family of simple functions $\{f_\Delta(x);\ \Delta \in \boldsymbol{\Delta}\}$ which converges to $f(x)$ uniformly on $E\backslash E(\infty)$ in the wider sense. Here $\boldsymbol{\Delta}$ denotes the direct set of all divisions of E.

Now, when we have the Moore-Smith limit

$$I = \lim_{\Delta} \int_E f_\Delta(x)dx,$$

we say that this limit is the **Riemann integral** of the function $f(x)$ on E and denote it as

$$I = \int_E f(x)dx.$$

This value I does not depend on the choice of a direct family of simple functions $\{f_\Delta(x);\ \Delta \in \boldsymbol{\Delta}\}$ which converges to $f(x)$ uniformly on $E\backslash E(\infty)$ in the wider sense.

We also say that the Riemann integral of $f(x)$ is the **multi-integral** or the **d-dimensional integral**.

In an intuitive sense, the Moore-Smith limit is considered as the limit obtained by the infinite iteration of the divisions of $E\backslash E(\infty)$ so that $E\backslash E(\infty)$ is divided as being finer, more finer and infinitely finer.

Remark 2.1 In the definition of the Riemann integral of a general measurable function $f(x)$ in the above, we define the Riemann integral as the Moore-Smith limit

$$\int_E f(x)dx = \lim_{\Delta} \int_E f_\Delta(x)dx$$

for a direct family of simple functions $\{f_\Delta(x);\ \Delta \in \boldsymbol{\Delta}\}$ which converges to $f(x)$ uniformly on $E \backslash E(\infty)$ in the wider sense.

Then, as for this Moore-Smith limit, we have either one of the following (1) and (2):

(1) It **converges**. (2) It **diverges**.

Further, in the case of convergence, we have either one of the following (3) and (4):

(3) It **converges absolutely**. (4) It **converges conditionally**.

(1) or (2) mean that we have the Moore-Smith limit or not. As for the case (3) or (4), we consider it as follows. The Moore-Smith limit **converges absolutely** if this Moore-Smith limit does not depend on the choice of the direct family of simple functions $\{f_\Delta(x);\ \Delta \in \boldsymbol{\Delta}\}$ which approximates $f(x)$. The Moore-Smith limit **converges conditionally** if this Moore-Smith limit depends on the choice of the direct family of simple functions $\{f_\Delta(x);\ \Delta \in \boldsymbol{\Delta}\}$ which approximates $f(x)$.

Further, we have the similar remarks for the convergence or the divergence of the Riemann integral of a simple function.

Here we consider the Riemann integral is the narrow sense which is studied usually until now.

Especially, we assume that E is a measurable bounded closed set and $f(x)$ is a bounded measurable function defined on E. Then we define the finite division of E as follows:

$$(\Delta):\ E = E_1 + E_2 + \cdots + E_n.$$

Then we consider a simple function

$$f_\Delta(x) = \sum_{j=1}^{n} f(\xi_j)\chi_{E_i}(x),\ (\xi_j \in E_j,\ (1 \le j \le n)).$$

Then the Riemann integral $f(x)$ is equal to the Moore-Smith limit

$$I = \int_E f(x)dx = \lim_{\Delta} \int_E f_\Delta(x)dx = \lim_{\Delta} \sum_{j=1}^{n} f(\xi_j)\mu(E_j) \tag{2.4}$$

if it exists. Namely, the Riemann integral of $f(x)$ is defined as the Moore-Smith limit of the **Riemann sum**

$$R_\Delta = \sum_{j=1}^{n} f(\xi_j)\mu(E_j).$$

This is the definition of the Riemann integral which is defined by the similar method to Riemann himself

Especially, when E is equal to a closed interval

$$K = [a_1,\ b_1] \times [a_2,\ b_2] \times \cdots \times [a_d,\ b_d],$$

we represent this Riemann integral as

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_d}^{b_d} f(x_1,\ x_2,\ \cdots,\ x_d)dx_1dx_2\cdots dx_d. \tag{2.5}$$

In the formula (2.4), we say that E is the **domain of integration**, $f(x)$ is the **integrand** and x is the **variables of integration**.

Then the integral I is independent of the variables of integration.

We say that $f(x)$ is **integrable** on E if the limit (2.4) exists. The integrable function is assumed to be a measurable function

Next we study the Darboux's theorem and the conditions of integrability of a measurable function.

Now we assume that E is a measurable bounded closed set in $\boldsymbol{R}^d$. Further we assume that a function $f(x)$ is a bounded measurable function on E.

A finite division of the set E is defined as follows:

$$(\Delta):\ E = E_1 + E_2 + \cdots + E_n.$$

Let M_i and m_i be the supremum and the infimum of $f(x)$ on the subset E_i respectively, $(i = 1,\ 2,\ \cdots, n)$. Let M and m be the supremum and the infimum of $f(x)$ on the set E respectively. Then, if we put

$$g_\Delta(x) = \sum_{i=1}^{n} m_i\chi_{E_i}(x), \tag{2.6}$$

$$h_\Delta(x) = \sum_{i=1}^{n} M_i\chi_{E_i}(x), \tag{2.7}$$

we have the Moore-Smith limits

$$\lim_{\Delta} g_\Delta(x) = f(x), \tag{2.8}$$

$$\lim_{\Delta} h_\Delta(x) = f(x) \tag{2.9}$$

which converge uniformly on E. Now let s_Δ and S_Δ be the Riemann integrals of $g_\Delta(x)$ and $h_\Delta(x)$ on E respectively. Namely we have the equalities

$$s_\Delta = \int_E g_\Delta(x)dx = \sum_{i=1}^{n} m_i \mu(E_i), \tag{2.10}$$

$$S_\Delta = \int_E h_\Delta(x)dx = \sum_{i=1}^{n} M_i \mu(E_i). \tag{2.11}$$

Here $\mu(E_i)$ denotes the Jordan measure of E_i, $(i = 1,\ 2,\ \cdots, n)$. Then we have the inqualities

$$m\mu(E) \leq s_\Delta \leq S_\Delta \leq M\mu(E). \tag{2.12}$$

Here, since $\mu(E) < \infty$ holds, $\{s_\Delta\}$ and $\{S_\Delta\}$ for all the divisions Δ of E are bounded. Therefore we have

$$S = \inf_\Delta S_\Delta,\ s = \sup_\Delta s_\Delta. \tag{2.13}$$

Then we have the inequality

$$s \leq S. \tag{2.14}$$

Thus we have the following Theorem 2.1.

Theorem 2.1(Darboux's theorem) *We use the notation in the above. Then we have the equalities*

$$\lim_\Delta s_\Delta = s,\ \lim_\Delta S_\Delta = S \tag{2.15}$$

in the sense of the Moore-Smith *limit.*

In the following, we study the integrability conditions of a function $f(x)$.

Theorem 2.2 *Assume that a set E is a measurable bounded closed set in $\boldsymbol{R}^d$ and $f(x)$ is a bounded measurable function on E. Then $f(x)$ is integrable on E if and only if $s = S$ holds.*

We use the notation in the above.

Then we say that $v_i = M_i - m_i$ is the **oscilletion** of $f(x)$ on the subset E_i of the division Δ, $(i = 1,\ 2,\ \cdots,\ n)$. Then, because we have

$$V_\Delta = S_\Delta - s_\Delta = \sum_{i=1}^{n} v_i \mu(E_i),$$

the condition $s = S$ is equivalent to the condition

$$\lim_\Delta V_\Delta = \lim_\Delta \sum_{i=1}^{n} v_i \mu(E_i) = 0$$

in the sense of the Moore-Smith limit. Hence we have the following Theorem 2.3.

Theorem 2.3 *Assume that E and $f(x)$ are the same as in Theorem 2.2. Then $f(x)$ is integrable if and only if we have the equality*

$$\lim_{\Delta} V_{\Delta} = \lim_{\Delta} \sum_{i=1}^{n} v_i \mu(E_i) = 0$$

in the sense of the Moore-Smith *limit.*

Corollary 2.1 *Assume that E and $f(x)$ are the same as in Theorem 2.3. Then we have the equality*

$$\lim_{\Delta} V_{\Delta} = \lim_{\Delta} \sum_{i=1}^{n} v_i \mu(E_i) = 0$$

if and only if, for an arbitrary real number $\varepsilon > 0$, there exists a certain division Δ of E such that we have

$$V_{\Delta} = \sum_{i=1}^{n} v_i \mu(E_i) < \varepsilon.$$

Proposition 2.1 *If a function $f(x)$ is integrable on a measurable bounded closed set E in $\boldsymbol{R}^d$, $f(x)$ is also integrable on an arbitrary measurable subset E' of E.*

Here we have the following Theorem 2.4 as the existence theorem of the Riemann integral.

Theorem 2.4 *A continuous function on a measurable bounded closed set E in $\boldsymbol{R}^d$ is integrable.*

Proposition 2.2 *If the set B of the discontinuous points of a bounded measurable function $f(x)$ on a measurable bounded closed set E in $\boldsymbol{R}^d$ has the Jordan measure 0, $f(x)$ is integrable on E.*

3 Fundamental properties of the Riemann integral

In this section, we study the fundamental properties of the Riemann integral.

Theorem 3.1 *Assume that a set E is a measurable set in $\boldsymbol{R}^d$ and a functions $f(x)$ is integrable on E. If $E = E_1 + E_2$ is a direct sum decomposition of E, we have the equality*

$$\int_E f(x)dx = \int_{E_1} f(x)dx + \int_{E_2} f(x)dx.$$

Theorem 3.2 *Assume that a set E is a measurable set in $\boldsymbol{R}^d$ and two functions $f(x)$ and $g(x)$ are integrable on E. Then we have the following (1)∼(6):*

(1) *For two constants α and β, $\alpha f(x) + \beta g(x)$ is also integrable on E and we have the following equality*

$$\int_E \{\alpha f(x) + \beta g(x)\}dx = \alpha \int_E f(x)dx + \beta \int_E g(x)dx.$$

(2) *If $f(x) \geq 0$ holds, we have the inequality*

$$\int_E f(x)dx \geq 0.$$

Further, if $f(x)$ is continuous at an interior point $x_0 \in E$ and $f(x_0) > 0$ holds, we have the following inequality

$$\int_E f(x)dx > 0.$$

(3) *If $f(x) \geq g(x)$ holds, we have the inequality*

$$\int_E f(x)dx \geq \int_E g(x)dx.$$

Further, if $f(x)$ and $g(x)$ are continuous at an interior point x_0 of E and $f(x_0) > g(x_0)$ holds, we have the following inequality

$$\int_E f(x)dx > \int_E g(x)dx.$$

(4) *$|f(x)|$ is also integrable on E and we have the inequality*

$$|\int_E f(x)dx| \leq \int_E |f(x)|dx.$$

(5) *$f(x)g(x)$ is also integrable on E.*

(6) *If there exists a constant* k *such that* $|g(x)| \geq k > 0$ *holds on* E, $\dfrac{f(x)}{g(x)}$ *is also integrable on* E.

Theorem 3.3(The first mean value theorem) *Assume that a set* E *is a measurable bounded closed set in* $\boldsymbol{R}^d$ *and a function* $f(x)$ *is bounded and integrable on* E.

Let M *and* m *be the supremum and the infimum of* $f(x)$ *on* E *respectively. Then there exists a certain real number* α *such that we have the equality*

$$\int_E f(x)dx = \alpha\mu(E), \ (m \leq \alpha \leq M).$$

Especially, if $f(x)$ *is continuous on the measurable bounded closed domain* E, *there exists a certain* ξ *in* E *such that* $\alpha = f(\xi)$ *holds.*

Extending Theorem 3.3, we have the mean value theorem of the following type.

Theorem 3.4(The first mean value theorem) *Assume that we have the following* (i)∼(iii):

(i) E *is a measurable bounded closed domain in* $\boldsymbol{R}^d$.

(ii) *A function* $f(x)$ *is continuous on* E.

(iii) *A function* $\varphi(x)$ *is bounded and integrable on* E *and* $\varphi(x)$ *has the definite sign in the weak sense.*

Then we have the equality

$$\int_E f(x)\varphi(x)dx = f(\xi)\int_E \varphi(x)dx$$

for a certain point $\xi \in E$.

Here we give the lemma which is the result necessary for the proof of the theorem in the above.

Lemma 3.1 *Assume that a set* E *is a measurable bounded closed domain and a function* $f(x)$ *is bounded and integrable on* E.

Then there exists at least one continuous point of $f(x)$ *in* E. *Therefore there exist the continuous points of* $f(x)$ *densely in* E.

Theorem 3.5 *Assume that a set* E *is a bounded measurable set in* $\boldsymbol{R}^d$ *and* $K = [a_1,\ b_1] \times [a_2,\ b_2] \times \cdots \times [a_d,\ b_d]$ *is a bounded closed interval including* E. *Assume that a function* $f(x)$ *is bounded and integrable on* E.

Put

$$f^*(x) = \begin{cases} f(x), & (x \in E), \\ 0, & (x \in K \backslash E) \end{cases}$$

formally. Then we have the following equality

$$\int_E f(x)dx = \int_K f(x)\chi_E(x)dx.$$

4 Iterated integral

In this section, we study the iterated integral.

In order to calculate the d-dimensional integral actually, it is suitable to calculate it by the iteration of the 1-dimensional integrals. In the following, we study this iterated integral.

At first, we prepare the notation.

We denote a variable $x = (x_1,\ x_2,\ \cdots,\ x_d)$ as $x = (x_1,\ x'),\ x' = (x_2,\ \cdots,\ x_d)$ and we denote $f(x) = f(x_1,\ x')$.

Theorem 4.1 *Assume that a function $f(x)$ is bounded and integrable on a bounded closed interval*

$$K = \prod_{i=1}^{d} [a_i,\ b_i] = \{x = (x_1,\ x_2,\ \cdots,\ x_d);\ a_i \le x_i \le b_i,\ (1 \le i \le d)\}.$$

If we put

$$K' = \prod_{i=2}^{d} [a_i,\ b_i],$$

we have the following (1) *and* (2):

(1) *If there exists*

$$\int_{K'} f(x_1,\ x')$$

for an arbitrary $x_1 \in [a_1,\ b_1]$, we have the following equality

$$\int_K f(x)dx = \int_{a_1}^{b_1} \{\int_{K'} f(x_1,\ x')dx'\}dx_1. \tag{4.1}$$

(2) *If there exists*

$$\int_{a_1}^{b_1} f(x_1,\ x')dx_1$$

for an arbitrary $x' \in K'$, *we have the following equality*

$$\int_K f(x)dx = \int_{K'} \{\int_{a_1}^{b_1} f(x_1,\ x')dx_1\}dx'. \tag{4.2}$$

Further we can study the $(d-1)$-dimensional integral with respect to x' in the similar way. This integral comes to the iterations of the 1-dimensional integrals inductively.

Further, we consider the section with respect to the other coordinate x_i of the variable x by the same method.

Thereby we have the following Corollary.

Corollary 4.1 *If a function* $f(x)$ *is continuous on a bounded closed interval*

$$K = \prod_{i=1}^{n} [a_i,\ b_i],$$

we have the following equality

$$\int_K f(x)dx = \int_{a_{i_1}}^{b_{i_1}} dx_{i_1} \int_{a_{i_2}}^{b_{i_2}} dx_{i_2} \cdots \int_{a_{i_d}}^{b_{i_d}} f(x_1,\ x_2,\ \cdots,\ x_d)dx_{i_d}.$$

Here $(i_1,\ i_2,\ \cdots,\ i_d)$ *is an arbitrary permutation of* $(1,\ 2,\ \cdots, n)$.

Theorem 4.2 *Let* E *be a measurable bounded closed set in* $\boldsymbol{R}^d$ *and let*

$$K = \prod_{i=1}^{d} [a_i,\ b_i]$$

be an arbitrary closed interval including E. *Let* $\chi_E(x)$ *be the defining function of* E. *If a function* $f(x)$ *is bounded and integrable on* E, *we put*

$$f^*(x) = \begin{cases} f(x), & (x \in E), \\ 0, & (x \in K \backslash E) \end{cases}$$

and we denote it as $f^*(x) = f(x)\chi_E(x)$ *formally. Then we have the following equality*

$$\int_E f(x)dx = \int_K f(x)\chi_E(x)dx.$$

Theorem 4.3 *Assume that E' is a measurable bounded closed set in $\boldsymbol{R}^{d-1}$ and two functions $\varphi_1(x')$ and $\varphi_2(x')$ are continuous on E' such as $\varphi_1(x') \le \varphi_2(x')$ holds for $x' \in E'$. Then the bounded closed set*

$$E = \{x = (x_1,\ x');\ x' \in E',\ \varphi_1(x') \le x_1 \le \varphi_2(x')\}$$

is measurable and its Jordan measure is equal to

$$\mu(E) = \int_{E'} \{\varphi_2(x') - \varphi_1(x')\} dx'.$$

Theorem 4.4 *Assume that $E,\ E',\ \varphi_1$ and φ_2 are the same as in Theorem 4.3. If a function $f(x)$ is continuous on E, we have the following equality*

$$\int_E f(x)dx = \int_{E'} dx' \int_{\varphi_1(x')}^{\varphi_2(x')} f(x_1,\ x')dx_1.$$

Theorem 4.5 *Assume that E is a measurable bounded closed set in $\boldsymbol{R}^d$ and*

$$K = \prod_{i=1}^{d} [a_i,\ b_i]$$

is an arbitrary bounded closed interval including E. For $1 \le i \le d,\ E(x_i)$ be the section of x by the hyperplane which is orthogonal to the x_i-axis through $x_i \in [a_i,\ b_i]$. Then we assume that $E(x_i)$ is the measurable bounded closed set in $\boldsymbol{R}^{d-1}$ for all $x_i \in [a_i,\ b_i]$. Then, if a function $f(x)$ is continuous on E, we have the following equality

$$\int_E f(x)dx = \int_{a_i}^{b_i} dx_i \int_{E(x_i)} f(x_1,\ x_2, \cdots,\ x_d) d\check{x}_i.$$

Here we denote $\check{x}_i = (x_1,\ \cdots,\ x_{i-1},\ x_{i+1},\ \cdots,\ x_d)$.

5 Method of calculation of the Riemann integral

In this section, we study one method of calculation of the Riemann integral. This is the method of approximating the domain of integration of the Riemann integral by using the direct family of measurable bounded closed sets. Until now, this is considered as the improper Riemann integral. The new definition of the Riemann integral includes the case of the improper Riemann integral already. The remarkable point is the method of calculation of the Riemann integral.

Assume that the domain of integration E is a general measurable set. Then the Jordan measure of the boundary $\dot{E}$ of E is 0.

Now we define that a direct family $\{E_\alpha; \alpha \in A\}$ of the measurable bounded closed sets included in E **converges** to E if, for an arbitrary measurable bounded closed set K included in E, there exists a certain $\alpha_0 \in A$ such that $K \subset E_\alpha$ holds for $\alpha \geq \alpha_0$.

Then we say that the direct family $\{E_\alpha\}$ is an **approximating direct family** of E.

Especially, put $A = \{1,\ 2,\ 3,\ \cdots\}$. Then we say that a sequence $\{E_n\}$ **monotonously converges** to E if $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$ holds.

In general, if a sequence $\{E_n\}$ converges to E and we put $E_1 \cup E_2 \cup \cdots \cup E_n = H_n,\ (n = 1,\ 2.\ 3,\ \cdots)$, then the sequence $\{H_n\}$ converges to E monotonously.

Assume that an integrand $f(x)$ is a measurable function on E. When we put $E(\infty) = \{x \in E;\ |f(x)| = \infty\}$, we say that a point in $E(\infty)$ is a **singular point** of $f(x)$. Then we assume that $E(\infty) \in \mathcal{B}$ holds and it has the Jordan measure 0. Thus $E \backslash E(\infty)$ is also a measurable set.

Further we assume that $f(x)$ is bounded and integrable on an arbitrary measurable bounded closed set in $E \backslash E(\infty)$. Then when the direct family $\{I(E_\alpha)\}$ defined by the formula

$$I(E_\alpha) = \int_{E_\alpha} f(x)dx$$

for an approximating direct family $\{E_\alpha\}$ of $E \backslash E(\infty)$ converges, we have the equality

$$\int_E f(x)dx = \lim_\alpha \int_{E_\alpha} f(x)dx$$

for the Moore-Smith limit

$$I = \lim_\alpha I(E_\alpha).$$

Then the Moore-Smith limit $\lim_\alpha I(E_\alpha)$ does not depend on the choice of the approximating direct family $\{E_\alpha\}$ of $E \backslash E(\infty)$ if the Riemann integral converges absolutely. Further the Moore-Smith limit in the above takes the various value of the integral depending on the choice of the approximating direct family $\{E_\alpha\}$ if the Riemann integral converges conditionally. This Moore-Smith limit does not exist if the Riemann integral diverges.

Remark 5.1 Until now, we say that the Riemann integral calculated by using the approximating direct family $\{E_\alpha\}$ as in the above is the **improper Riemann integral**. Nevertheless, we see that the definition of the Riemann integral in this paper includes the definition of the improper Riemann integral.

Now, for a measurable function $f(x)$ on E, we put

$$f^+(x) = \sup(f(x),\ 0),\ f^-(x) = -\inf(f(x),\ 0).$$

Then we have the following relations

$$|f(x)| \geq f^+(x) \geq 0,\ |f(x)| \geq f^-(x) \geq 0,$$
$$f(x) = f^+(x) - f^-(x),\ |f(x)| = f^+(x) + f^-(x).$$

Hence $f(x)$ is integrable if and only if $f^+(x)$ and $f^-(x)$ are integrable. Then we have the equality

$$\int_E f(x)dx = \int_E f^+(x)dx - \int_E f^-(x)dx.$$

Here we have the relations concerning the convergence and divergence of the Riemann integral of $f(x)$ on E in the following table.

Table 5.1 Convergence and divergence of the Riemann integral

(conv.=convergence, div.=divergence, abs.conv.=absolute convergence, cond.conv.=conditional convergence.)

$\int_E f(x)dx$	$\int_E \lvert f(x)\rvert dx$	$\int_E f^+(x)dx$	$\int_E f^-(x)dx$
abs.conv.	conv.	conv.	conv.
div.	div.	conv.	div.
div.	div.	div.	conv.
cond.conv. or div.	div.	div.	div.

We have the following properties of the Riemann integral.

Theorem 5.1 *Assume that E is a measurable subset of $\boldsymbol{R}^d$. Assume that a function $f(x)$ is measurable and nonnegative on E and the set $E(\infty)$ of its singular points has the Jordan measure* 0. *Then if $\lim_\alpha I(E_\alpha)$ exists for a certain approximating direct family $\{E_\alpha\}$ of $E \backslash E(\infty)$, the Riemann integral of $f(x)$ on E converges absolutely.*

Theorem 5.2 *Let E, $f(x)$ and $E(\infty)$ be the same as in Theorem* 5.1. *Then*

$$\int_E f(x)dx$$

converges absolutely if and only if

$$I(H) = \int_H f(x)dx$$

is bounded for all measurable bounded closed set H *included in* $E\backslash E(\infty)$.

Theorem 5.3 *Assume that* E *is a measurable bounded set in* $\boldsymbol{R}^d$ *and a function* $f(x)$ *is Riemann integrable on* E. *Further assume that* $f(x) \geq 0$ *holds. Assume that a sequence* $\{E_n\}$ *of measurable subsets of* E *satisfies the following conditions* (1) *and* (2):

(1) $f(x)$ *is Riemann integrable on* E_n, $(n \geq 1)$.

(2) *We have* $\mu(E\backslash E_n) \to 0$, $(n \to \infty)$.

Then we have the equality

$$\lim_{n\to\infty} \int_{E_n} f(x)dx = \int_E f(x)dx. \tag{5.1}$$

Remark 5.2 If a function $f(x)$ is not Riemann integral on E in Theorem 5.3, we have

$$\int_E f(x)dx = +\infty.$$

Then, if the other conditions are the same as in Theorem 5.3, the equality (5.1) holds in the sense that the left hand side of the formula (5.1) diverges to $+\infty$.

Proposition 5.1 *Let* E *be a measurable set in* $\boldsymbol{R}^d$. *Let* $f(x)$ *be a extended real valued measurable function on* E. *Further assume that* $E(\infty)$ *is the set of the singular points of* $f(x)$ *and* $\mu(E(\infty)) = 0$ *holds. Assume that a direct family* $\{f_\Delta(x)\}$ *of simple functions converges to* $f(x)$ *uniformly on* $E\backslash E(\infty)$ *in the wider sense and a direct family* $\{E_\alpha\}$ *of measurable bounded closed sets is an approximating direct family of* $E\backslash E(\infty)$. *Then we consider two equalities of the Riemann integral as follows*:

(I) $\displaystyle\int_E f(x)dx = \lim_\Delta \int_E f_\Delta(x)dx.$

(II) $\displaystyle\int_E f(x)dx = \lim_\alpha \int_{E_\alpha} f(x)dx.$

Then the convergence or the divergence of the Riemann integrals in (I) *or* (II) *are equivalent. Further the absolute convergence or the conditional convergence of the Riemann integrals* (I) *or* (II) *are equivalent.*

6 Change of variables

In this section, we prove the formula of the change of variables of the d-dimensional integral.

Now we assume that a C^1-mapping $x = \varphi(u)$ from an open set U' in u-space onto an open set U in x-space one to one. Then if we have

$$J = \frac{\partial(x)}{\partial(u)} = \frac{\partial(x_1,\ x_2,\ \cdots,\ x_d)}{\partial(u_1,\ u_2,\ \cdots,\ u_d)} \neq 0$$

on U', the inverse mapping $u = \varphi^{-1}(x)$ is C^1-class on U. Assume that the closure of the domain of integration is a bounded set included in the open set U, and E and E' correspond one to one by the mapping φ. Then the closure of E' is a bounded set included in U' and the boundary $\dot{E}$ of E and the boundary $\dot{E}'$ of E' correspond one to one.

Then we have the following Theorem 6.1.

Theorem 6.1 *Assume that a C^1-mapping $x = \varphi(u)$ maps an open set U' in u-space onto an open set U in x-space one to one and we have $J = \dfrac{\partial(x)}{\partial(u)} \neq 0$. Assume that E' is a bounded measurable set such that $\overline{E'}$ is included in U'.*

Put $\varphi(E') = E$. Then E is a bounded measurable set such that $\overline{E}$ is included in U and we have the equality

$$\mu(E) = \int_{E'} |\frac{\partial(x)}{\partial(u)}|du.$$

Theorem 6.2 *We use the same notation as in Theorem 6.1. Then, if $f(x)$ is bounded and integrable on E, we have the following formula of the change of variables*

$$\int_E f(x)dx = \int_{E'} f(\varphi(u))|\frac{\partial(x)}{\partial(u)}|du.$$

Theorem 6.3 *Assume that U' and U are both the bounded measurable open sets and a C^1-mapping $x = \varphi(u)$ maps U' onto U one to one. Further assume that $J = \dfrac{\partial(x)}{\partial(u)} \neq 0$ holds on U' and J is bounded. Then, if $f(x)$ is bounded and integrable on U, we have the following formula of the change of variables*

$$\int_E f(x)dx = \int_{U'} f(\varphi(u))|\frac{\partial(x)}{\partial(u)}|du.$$

Theorem 6.4 *Assume that U' and U are two measurable subsets in u-space and x-space respectively which are not necessarily bounded. Further, assume that a C^1-mapping $x = \varphi(u)$ maps U' onto U. Then put*

$$N' = \{u \in U';\ J(u) = 0\}$$

for $J = \dfrac{\partial(x)}{\partial(u)}$ and put $N = \varphi(N')$. Then assume that the Jordan measures of N and N' are 0 and $U' \backslash N'$ is mapped onto $U \backslash N$ one to one. Here let S be the set of the singular points of a function $f(x)$ and put $\varphi^{-1}(S) = S'$. Here assume that the Jordan measures of S and S' are 0. Then, if a function $f(x)$ is Riemann integrable on U, we have the following formula of the change of variables

$$\int_U f(x)dx = \int_{U'} f(\varphi(u))|J(u)|du.$$

Example 6.1(Polar coordinates of $\boldsymbol{R}^3$) The polar coordinates $(r,\ \theta,\ \varphi)$ of a point $P(x,\ y,\ z)$ of 3-dimensional Euclidean space $\boldsymbol{R}^3$ is defined by the C^1-mapping

$$\Phi:\ x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta,$$

$$(r \geq 0,\ 0 \leq \theta \leq \pi,\ 0 \leq \varphi < 2\pi)$$

from $r\theta\varphi$-space to xyz-space. Then assume that E' is a measurable subset of $r\theta\varphi$-space and put $\Phi(E') = E$. If a function $f(x,\ y,\ z)$ is integrable on E, we have the following formula

$$\begin{aligned}&\int_E f(x,\ y,\ z)dxdydz\\ =&\int_{E'} f(r\sin\theta\cos\varphi,\ r\sin\theta\sin\varphi,\ r\cos\theta)r^2\sin\theta drd\theta d\varphi.\end{aligned}$$

Example 6.2(Polar coordinates of $\boldsymbol{R}^d$) The polar coordinates $(r,\ \theta_1,\ \theta_2,\ \cdots,\ \theta_{d-1})$ of a point $P(x_1,\ x_2,\ \cdots,\ x_d)$ of d-dimensional Euclidean space is defined by the C^1-mapping

$$\Phi:\ \begin{cases} x_1 = r\cos\theta_1, \\ x_2 = r\sin\theta_1\cos\theta_2, \\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{d-1} = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-2}\cos\theta_{d-1}, \\ x_d = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-2}\sin\theta_{d-1}, \end{cases}$$

$$(r \geq 0,\ 0 \leq \theta_1,\ \theta_2,\ \cdots,\ \theta_{d-2} \leq \pi,\ 0 \leq \theta_{d-1} < 2\pi)$$

from $r_1\theta_1 \cdots \theta_{d-1}$-space to $x_1x_2 \cdots x_d$-space. Then assume that E' is a measurable subset of $r\theta_1 \cdots \theta_{d-1}$-space and put $\Phi(E') = E$. Then, if a function $f(x)$ is integrable on E, we have the following formula of the change of variables

$$\begin{aligned}
&\int_E f(x_1,\ x_2,\ \cdots,\ x_d) dx_1 dx_2 \cdots dx_d \\
= &\int_{E'} f(r\cos\theta_1,\ r\sin\theta_1\cos\theta_2,\ \cdots, \\
&r\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-2}\cos\theta_{d-1},\ r\sin\theta_1\sin\theta_2\cdots \\
&\sin\theta_{d-2}\sin\theta_{d-1}) r^{d-1}(\sin\theta_1)^{d-2}(\sin\theta_2)^{d-3}\cdots \\
&\sin\theta_{d-2} dr d\theta_1 d\theta_2 \cdots d\theta_{d-2} d\theta_{d-1}.
\end{aligned}$$

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