On Restricted Wythoff’s Nim

By

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Abstract

We shall study the following restricted Wythoff’s Nim. Let $S_i$ ($1 \leq i \leq 3$) be the set of positive integers. Each player can remove the number of tokens $s_1 \in S_1$ from the first pile and $s_2 \in S_2$ from the second pile and remove the same number of tokens $s_3 \in S_3$ from both piles. We shall identify $(m, n)$ to a position of this nim, where $m$ is the number of tokens in the first pile and $n$ is the number of tokens in the second pile. In the case $|S_2| < \infty$, we will show the Sprague-Grundy sequence(or simply G-sequences) $g_S(m, n)$ is periodic for fixed $m$.

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1 Introduction

In his paper [1], C. L. Bouton introduced the 2-player impartial combinatorial game, which is now called nim game. In [6], W. A. Wythoff modified the rule...
of this game as follows. The game is played by two players. There are two piles of tokens (or stones). Two players play alternately and either take from one of the piles an arbitrary number of tokens or from both piles of the same number of tokens. The player who takes up the last token is the winner. A position from which the player who made the last move, the previous player, can always win is called a P-position. The P-positions of original nim of Bouton are \((k, k)\) with arbitrary \(k \geq 0\) and the following P-positions of Wythoff’s nim are related to the golden ratio.

**Proposition 1.1** Wythoff (1905)

\((m, n)\) is a P-position \( \iff \) \((m, n) = (m_s, m_s + s)\) or \((m_s + s, m_s)\), where \(m_s\) is determined by \(m_s = [s\phi]\). Here \(\phi = \frac{1 + \sqrt{5}}{2}\).

In this paper, we shall consider some restricted Wythoff’s nim as follows. Let \(S_i\) \((1 \leq i \leq 3)\) be the set of positive integers. Each player can remove the number of tokens \(s_1 \in S_1\) from the first pile and \(s_2 \in S_2\) from the second pile and remove the same number of tokens \(s_3 \in S_3\) from both piles. We shall identify \((m, n)\) to a position of this Nim, where \(m\) is the number of tokens in the first pile and \(n\) is the number of tokens in the second pile. Assume the set of positive integers \(S_2\) is finite. In the next section, we shall show the Sprague-Grundy sequence \(g_S(m, n)\) is periodic for fixed \(m\), i.e., there exist \(a_m \geq 0\) and \(p_m > 0\) such that \(g_S(m, n + p_m) = g_S(m, n)\) for any \(n \geq a_m\). Here \(p_m\) is called the period of this Sprague-Grundy sequence \(g_S(m, n)\).

### 2 Proof of Main Theorem

Let \(S_2 = \{s_1, s_2, \ldots, s_{s(2)} \mid 0 < s_1 < s_2 < \cdots < s_{s(2)}\}\) be the set of positive integers. The player is restricted to remove the number of tokens \(s \in S_2\) from the second pile.

**Theorem 2.1** Under the above notations, \(g_S(m, n)\) has a period \(p_m\) for any fixed \(m\), that is,

\[n \geq a_m \implies g_S(m, n + p_m) = g_S(m, n), \text{ for any } n \geq a_m.\]

Proof. From the assumption on \(S_2\), G-sequence satisfies \(0 \leq g_S(m, n) \leq 2m + s(2)\). The case \(m = 0\) is nothing but the case of restricted one pile nim and it is known that \(g_S(0, n)\) is periodic. Thus, assume \(g_S(m', n + p_m') = g(m', n)\) for any \(n \geq a_m'\) for the cases \(m'(0 \leq m' \leq m - 1)\). \(p_0, p_1, \ldots, p_{m-1}\) denote the periods for the cases \(0 \leq m' \leq m - 1\). Put \(a_s = \max\{a_0, a_1, \ldots, a_{m-1}\}\), \(p_s = \text{LCM}(p_0, p_1, \ldots, p_{m-1})\). Then the pigeonhole principle asserts that there exists a period \(p = p_m\) \((p | p_m)\) as follows.
The number of patterns of consecutive $s(2)$ Grundy numbers $g_S(m, n)$ are at most $\ell_* = (2m + s(2) + 1)^{s(2)} + 1$ pairs;

\[
\{ (g_S(m, a_*), g_S(m, a_* + 1), \ldots, g_S(m, a_* + s(2) - 1)),
    (g_S(m, a_* + p_*), g_S(m, a_* + p_* + 1), \ldots, g_S(m, a_* + p_* + s(2) - 1)), 
    \ldots 
    (g_S(m, a_* + \ell_*p_*), g_S(m, a_* + \ell_*p_* + 1), \ldots, g_S(m, a_* + \ell_*p_* + s(2) - 1)) \}.
\]

Hence the pigeonhole principle asserts that there exists a pair $\ell_i, \ell_j$ ($0 \leq \ell_i < \ell_j \leq \ell_*$) which satisfies

\[
(\ell_i, \ell_j) \equiv (\ell_j, \ell_i) \pmod{\ell_*}, \quad 0 \leq \ell_i < \ell_j \leq \ell_*,
\]

Putting $\rho = a_* + p_*\ell_i$ and $p = p_*(\ell_j - \ell_i)$. From the above condition,

\[
\begin{align*}
    g_S(m, a) &= g_S(m, a + p) \\
    \vdots &= \vdots \\
    g_S(m, a + s(2) - 1) &= g_S(m, a + s(2) - 1 + p)
\end{align*}
\]

Thus $g_S(m, n)$ has period $p$ for $a \leq n \leq a + s(2) - 1$. Assume $g_S(m, n') = g_S(m, n' + p)$ for any $n'$ ($a \leq n' < n$). Since $n - s_j \geq a, n - s_k \geq a, g_S(m, n + p) = \max\{g_S(m - s_i, n + p), g_S(m, n + p - s_j), g_S(m - s_k, n + p - s_k) \mid s_i, s_j, s_k \in S \text{ with } 0 \leq m - s_i \text{ and } 0 \leq m - s_k \} = \max\{g_S(m - s_i, n), g_S(m, n - s_j), g_S(m - s_k, n - s_k) \mid s_i, s_j, s_k \in S \text{ with } 0 \leq m - s_i \text{ and } 0 \leq m - s_k \} = g_S(m, n)$.

Hence, by induction, we have $n \geq a \implies g_S(m, n + p) = g_S(m, n)$.

Now we shall consider the special case when $|S_1|, |S_2|$ and $|S_3|$ are finite. Put $S_1 = \{s_{1,1}, s_{1,2}, \ldots, s_{1,(r(1)} \mid 0 < s_{1,1} < s_{1,2} < \cdots < s_{1,(r(1)} \}$, $S_2 = \{s_{2,1}, s_{2,2}, \ldots, s_{2,(r(2)} \mid 0 < s_{2,1} < s_{2,2} < \cdots < s_{2,(r(2)} \}$, and $S_3 = \{s_{3,1}, s_{3,2}, \ldots, s_{3,(r(3)} \mid 0 < s_{3,1} < s_{3,2} < \cdots < s_{3,(r(3)} \}$. $s(0)$ denotes $\max(s_{1,(r(1)}, s_{2,(r(2)}, s_{3,(r(3)} \)$. Assume that there exist a positive integer $p$ which satisfies $g_S(m, n + p) = g_S(m, n)$ for any $0 \leq m, n \leq s(0) + p$.

Then we have the following special case of the above theorem.

**Corollary 2.2** Under the above notation, $g_S(m, n)$ is purely periodic and satisfies $g_S(m, n + p) = g_S(m, n)$ for any $m, n \geq 0$.

Using this corollary, we have the following example.
Table 1 (The case $S = \{\{1,2\}, \{1,2\}, \{1,2\}\})$

<table>
<thead>
<tr>
<th>$m \setminus n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
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<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
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<td>1</td>
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<td>2</td>
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<td>0</td>
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<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Thus we have $g_S(m,n) \equiv g(m,n) \pmod{3}$ and $g_S(m,n + 3) = g_S(m,n)$ for any $m$ and $n$.

3 Equivalent classes of $S \subset \mathbb{N}$

We call $S \subset \mathbb{N}$ and $S' \subset \mathbb{N}$ is equivalent if and only if

$$S \sim S' \iff g_S(k) = g_{S'}(k) \text{ (for any } k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}).$$

For $n$ piles $(m_1,m_2,\ldots,m_n)$, the number of removable tokens from the pile $m_k$ is restricted $s \in S_k$ for each $S_k$ of $S = (S_1,S_2,\ldots,S_k,\ldots,S_n)$. Then we shall slightly generalizes the equivalent classes for $S = (S_1,S_2,\ldots,S_n)$, and $S' = (S'_1,S'_2,\ldots,S'_n)$,

$$S \sim S' \iff g_S(k_1,k_2,\ldots,k_n) = g_{S'}(k_1,k_2,\ldots,k_n) \text{ (for any } k_i \in \mathbb{N}_0 \text{ (}1 \leq i \leq n\text{)).}$$

Then nim-sum implies

$$S \sim S' \iff S_i \sim S'_i \text{ (}1 \leq i \leq n\text{)}.$$ 

For any $S_i, S'_i \subset \mathbb{N} \text{ (}1 \leq i \leq 3\text{)},$ we shall write $S = (S_1,S_2,S_3), S' = (S'_1,S'_2,S'_3)$ and consider the restricted Wythoff’s nim. In the case $S$, the number of removable tokens from the first pile is restricted to $s \in S_1$, the number of removable tokens from the second pile is restricted to $s \in S_2$, and the number of removable tokens from both piles at the same time is restricted to $s \in S_3$. The case $S'$ is same as the case $S$. Now we will consider two restricted Wythoff’s nim such as the number of tokens each player can remove from the piles are restricted $s \in S$ and $s \in S'$. Then for these restricted Wythoff’s nim, there exist several $S$ and $S'$ with $S \sim S'$ but $g_S(m,n) \neq g_{S'}(m,n)$ for some $(m,n)$.

Put $S = 2\mathbb{N}$ and $S' = \mathbb{N} \setminus \{1\}$. Then it is known that $S \sim S'$. We have calculated $g_S(m,n) g_{S'}(m,n)$ for small $(m,n)$ as follows.
Thus we have

\[ g(S) \triangleq \text{the number of tokens each player can remove from the piles are restricted to} \]

\[ S \sim \text{the number of tokens from the second pile is restricted to} \]

\[ S \sim \text{ tokens from both piles at the same time is restricted to} \]

Then nim-sum implies

\[ S \sim \text{ Put} \]

\[ \text{For any} \]

\[ S \sim \text{for each} \]

\[ \text{with} \]

\[ \text{Put} \]

\[ \text{For} \]

\[ \text{and} \]

\[ \text{Put} \]

\[ \text{Put} \]

\[ \text{Put} \]

\[ \text{Put} \]

\[ \text{Put} \]

\[ \]