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Commutative Nil-Neat Group Rings

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Abstract

Let R be a ring and G a group. We establish a rather curious necessary and sufficient condition for the commutative group ring RG to be *nil-neat* only in terms of R,G and their sections. This somewhat extends two recent results established by McGovern et al. in (J. Algebra Appl., 2015) and by Udar et al. in (Commun. Algebra, 2017), related to commutative nil-clean and neat group rings, respectively.

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1 Introduction and the main result

All given rings are assumed to be commutative rings with nonzero identity as well as all given groups are assumed to be multiplicative abelian. For such a ring R, as usual, $J(R)$ denotes the Jacobson radical, while for such a group

G, as a standard setting, G_p denotes the p-torsion component for some fixed prime integer p. We shall say that G is a p-group whenever $G = G_p$. Suppose RG is now the group ring of G over R.

A ring R is called *nil-clean* if all its elements are sums of an idempotent and nilpotent, that is, $R/J(R)$ is a boolean ring in the sense that each its element is an idempotent and $J(R)$ is nil in the sense that each its element is a nilpotent. It was obtained in [3] a criterion for RG to be nil-clean only in terms associated with G and R, namely it was proved that *RG is a nil-clean ring if, and only if, R is a nil-clean ring and G is a 2-group (i.e., $G = G_2$).*

Moreover, a ring R is said to be *neat* if every its proper homomorphic image is *clean* in the sense that each its element is a sum of an idempotent and a unit. Notice that nil-clean rings are always clean, whereas the latter ones are neat as these two implications are *not* reversible. In [5] were investigated commutative neat group rings (in parallel to commutative clean group rings studied by too many authors but without having a final satisfactory result yet – see the bibliography therewith), but unfortunately a final criterion was not found so far, too.

On the other hand, in [4] was discovered a new class of rings, termed *nil-neat rings* as those rings whose proper homomorphic images are nil-clean. It was obtained there that all nil-clean rings are nil-neat with wrong converse validity as the corresponding example in [4, Example 2.10] demonstrably shows, as well as that nil-neat rings are themselves clean with having again a nonreversible relationship.

So, the motivation of writing up this short note is to generalize the achievement from [3] to the larger class of nil-neat rings as well as to strengthen the achievement from [5] to this new point of view by obtaining a result presented in a final form. A crucial formula in our argumentation will be the following one due to Karpilovsky (see [2]): For a ring R and a group G, the next equality is true:

$$J(RG) = J(R)G + \langle r(g_p - 1) \mid r \in R, pr \in J(R), g_p \in G_p \rangle.$$

Specifically, the following main statement holds:

Theorem 1.1 *A commutative group ring RG is nil-neat if, and only if, exactly one of the next two conditions is valid:*

(1) $G = \{1\}$ and R is nil-neat;

(2) $G \neq \{1\}$ such that G is a 2-group and R is nil-clean.

Proof. If $G = \{1\}$, there is nothing to prove because $RG \cong R$, so we shall assume hereafter that G is possibly nontrivial. If now R is nil-clean and G is a 2-group, then by the utilization of [3, Theorem 2.6] we know that RG is nil-clean and thus, as noted above, it has to be nil-neat. So, we will be concentrated on the converse implication. To that goal, assuming that RG is nil-neat, as R is its proper homomorphic image, we detect that R is nil-clean. If R is not a field (and thus it is different to $\mathbb{Z}_2$), R possesses the simple field of two elements $\mathbb{Z}_2$ as a proper epimorphic image since $R/M \cong \mathbb{Z}_2$ for any maximal ideal M of R, whence one directly sees that the group ring $\mathbb{Z}_2G$ is a proper epimorphic image for RG. Therefore, $\mathbb{Z}_2G$ is nil-clean, and it follows from the aforementioned theorem from [3] that G is a 2-group, as required.

If now $R \cong \mathbb{Z}_2$, we consider two cases about the 2-torsion component G_2 of G as follows:

Case 1: $G_2 \neq \{1\}$. Hence $\mathbb{Z}_2(G/G_2)$ is nil-clean being a proper homomorphic image of $\mathbb{Z}_2G$. Therefore, the aforementioned result from [3] guaranteed that G/G_2 is a 2-group, so that it follows at once that $G = G_2$, as desired.

Case 2: $G_2 = \{1\}$. We claim that either the torsion subgroup is trivial, that is, $G_t = \coprod_{\forall p} G_p = \{1\}$, or that G is q-torsion, i.e., $G = G_q$ for some prime q with $(q, 2) = 1$. To see that, assume in a way of contradiction that the q-component of torsion G_q is not equal to $\{1\}$ for any prime q different to 2. Consequently, $\mathbb{Z}_2(G/G_q)$ is nil-clean as being a proper homomorphic image of $\mathbb{Z}_2G$. Thus, as already commented above, the quotient G/G_q is a 2-group and so it is readily checked that $G = G_q$ because each q-torsion component is 2-divisible, i.e., $(G_q)^2 = G_q$ (see [1]). If, however, there are two such different primes p, q with non-identity G_p, G_q, this will lead to $G = G_p = G_q$. But as $G_q \cap G_p = \{1\}$ for any two $p \neq q$, we derive that $G = \{1\}$, a contradiction, which substantiates our initial claim that G_t is trivial. If, however, only one such a prime q exists, we will just have that G is a q-group.

Furthermore, by virtue of the formula from [2] visualized above, one has in both cases (namely, when $G_t = \{1\}$ or when $G = G_q$) that $J(\mathbb{Z}_2G) = \{0\}$. Indeed, $J(\mathbb{Z}_2) = \{0\}$ and, for any $r \in \mathbb{Z}_2$, the relation $qr = 0$ implies that $r = 0$ as q is invertible in $\mathbb{Z}_2$.

Next, by using [4, Corollary 2.12], $\mathbb{Z}_2G$ must be either a field, which is pretty impossible as G is not $\{1\}$, or $\mathbb{Z}_2G$ is boolean being a subdirect product of a family of copies of the field $\mathbb{Z}_2$. That is why, in this case, for any g in $G \subseteq \mathbb{Z}_2G$, it must be that $g^2 = g$, whence $g = 1$, which is once again against our assumption that $G \neq \{1\}$. This means that the examined case is non-sense, and so we are set. $\square$

Remark. In order the given above proof to be more nearly self-contained and friendly to the reader, it is worthwhile to mention here that it follows directly from the manipulations made above that if $G_2 = \{1\}$, then $G = \{1\}$, provided RG is nil-neat (in particular, when it is nil-clean). So, surprisingly, RG can be a proper nil-neat ring precisely when G is the identity group and R is nil-neat but *not* nil-clean.

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