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Axiomatic Method of Measure and Integration (IV). Definition of the Lebesgue Integral and its Fundamental Properties

(Yoshifumi Ito "Differential and Integral Calculus II", Chapters 6, 7, 9)

By

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Abstract

In this paper, we define the Lebesgue integral of the Lebesgue measurable function on $\boldsymbol{R}^d$, $(d \geq 1)$.

Then we study the method of calculation of the Lebesgue integral. Further we clarify the convergence properties of the Lebesgue integral completely. These facts are the new results.

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Introduction

This paper is the part IV of the series of papers on the axiomatic method of measure and integration on the Euclidean space.

As for the details, we refer to Ito [12]. Further we refer to Ito [1] $\sim$ [11], [13] $\sim$ [19].

In this paper, we define the d-dimensional Lebesgue integral and study their fundamental properties. Here we assume $d \geq 1$.

We assume that the d-dimensional Euclidean space $\boldsymbol{R}^d$ is the Lebesgue measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$. Then we define the class of the Lebesgue measurable functions which adapt this Lebesgue measure. We define the Lebesgue integral of these Lebesgue measurable functions.

Then, in this paper, we define that a function $f(x)$ is Lebesgue measurable by the condition that it is a limit of a sequence of simple functions in the sense of pointwise convergence.

Here the convergence in the sense of pointwise convergence means the pointwise convergence on the set of points except all singular points of a function $f(x)$. As for details, we explain this in section 1.

This is the similar method to the method of defining a Jordan measurable function by the condition that it is a limit of a direct family of simple functions in the wider sense of uniform convergence in the theory of Riemann integral.

In this point, this method is different from the method of defining the Lebesgue integral until now.

Since the Lebesgue integral is defined for a Lebesgue measurable function, we have to prove that the Lebesgue measurability of functions is preserved for four fundamental rules of calculation and operations of taking the supremum, the infimum and the limit in order to study the relations between the Lebesgue integral and the operations of functions. We show these results as the theorems for the properties of Lebesgue measurable functions.

We define the Lebesgue integral for these Lebesgue measurable functions.

Assume that a simple function $f(x)$ on a measurable set E of $\boldsymbol{R}^d$ is defined as follows:

$$f(x) = \sum_{p=1}^{\infty} a_p \chi_{E_p}(x),\ (a_p \in \overline{\boldsymbol{R}},\ (1 \leq p < \infty))$$

for the countable division of E

$$E = \sum_{p=1}^{\infty} E_p,\ (E_p \in \mathcal{M},\ (1 \leq p < \infty)).$$

Then we define the Lebesgue integral of $f(x)$ as the sum of the series in the right hand side of the following formula

$$\int_E f(x)dx = \sum_{p=1}^{\infty} a_p \mu(E_p).$$

Here we assume that the series in the right hand side converges absolutely.

We define the Lebesgue integral of a general measurable function $f(x)$ on E by the formula

$$\int_E f(x)dx = \lim_{n\to\infty} \int_E f_n(x)dx.$$

Here we assume that a sequence $\{f_n(x)\}$ of the simple functions converges to $f(x)$ on E in the sense of pointwise convergence.

In general, the Lebesgue integral of $f(x)$ either converges or diverges. In the case of convergence, there is either one of the case of absolute convergence and the case of conditional convergence. In the case of conditional convergence, we say until now that the Lebesgue integral is the improper Lebesgue integral.

Then the concept of pointwise convergence well conforms to the class of all Lebesgue measurable functions and the class of Lebesgue integrable functions.

Namely the limit function of a sequence of functions in these classes in the sense of pointwise convergence belongs to the same class.

Thus, by using the similar expression to the theory of Riemann integral, it is seen clear that we have the difference that the convergence of a sequence of functions well confirms to the uniform convergence in the theory of Riemann integral and the convergence of a sequence of functions well confirms to the pointwise convergence in the theory of Lebesgue integral.

In the theory of Lebesgue integral, the reason why a integral domain E is a Lebesgue measurable set is the following.

If we assume that a considered subset E of $\boldsymbol{R}^d$ is not a Lebesgue measurable set, even a constant function on E is not a Lebesgue measurable function. After all, it is meaningless in itself to consider any Lebesgue measurable function on such a set E. Therefore the definition of the Lebesgue integral on a Lebesgue non-measurable set E is meaningless.

Thus it is meaningless to consider a Lebesgue non-measurable set and a Lebesgue non-measurable function in the theory of Lebesgue integral. These consideration is not the problem of the theory of Lebesgue integral.

Further, since the Lebesgue measure is a complete measure, we cannot consider the more extended measure theory including the Lebesgue non-measurable set by extending it. In this point, it is meaningless to consider the Lebesgue non-measurable sets.

In this paper, the range of a function is considered to be the subset of the space of the extended real numbers $\overline{\boldsymbol{R}} = [-\infty, \infty]$.

Here I show my heartfelt gratitude to my wife Mutuko for her help of typesetting this manuscript.

1 Lebesgue measurable functions

In this section, we define the concept of the Lebesgue measurable functions and study their fundamental properties. Here we assume $d \geq 1$.

Assume that the d-dimensional Euclidean space $\boldsymbol{R}^d$ is the d-dimensional Lebesgue measure space $(\boldsymbol{R}^d, \mathcal{M}, \mu)$.

Assume that a subset E of $\boldsymbol{R}^d$ is a Lebesgue measurable set. Hereafter, for simplicity, we say that E is measurable.

Now we consider a measurable function defined on a set E. Then we assume that all considered functions $f(x)$ are the extended real-valued functions defined on E. Namely we assume that the range of a function $f(x)$ is included in the space of the extended real numbers $\overline{\boldsymbol{R}} = [-\infty,\ \infty]$.

We denote the family of all Lebesgue measurable sets included in E as $\mathcal{M}_E$ and the restricted measure of the Lebesgue measure μ on $\boldsymbol{R}^d$ to the elements of $\mathcal{M}_E$ as μ. Then we say that the measure space $(E,\ \mathcal{M}_E,\ \mu)$ is the ***d*-dimensional Lebesgue measure space** on E. Hereafter we consider this Lebesgue measure space $(E,\ \mathcal{M}_E,\ \mu)$ when we study the Lebesgue integral of a Lebesgue measurable function $f(x)$ on E. Further we happen to denote $\mathcal{M}_E$ as $\mathcal{M}$ for simplicity.

At first, we define the concept of the simple functions.

Definition 1.1 We define that a function $f(x)$ defined on a measurable set E of $\boldsymbol{R}^d$ is a **simple function** if $f(x)$ is defined to be

$$f(x) = \sum_{p=1}^{\infty} a_p \chi_{E_p}(x),\ (x \in E) \tag{1.1}$$

for a countable division

$$(\Delta):\ E = \sum_{p=1}^{\infty} E_p = E_1 + E_2 + \cdots . \tag{1.2}$$

Here a_p is a real number or $\pm\infty$ for $1 \leq p < \infty$ and they are not necessarily different. $\chi_{E_p}(x)$ denotes the defining function of a set E_p, $(1 \leq p < \infty)$. Then we denote this simple function $f(x)$ as $f_\Delta(x)$. Here we assume that all the subsets $E_1,\ E_2,\ \cdots$ of E are the Lebesgue measurable sets and they are mutually disjoint. Further, we assume that $E(\infty) = \{x;\ |f(x)| = \infty\} \in \mathcal{M}$ holds and $\mu(E(\infty)) = 0$ holds.

In Definition 1.1, we define the defining function $\chi_A(x)$ of a set A as follows:

$$\chi_A(x) = \begin{cases} 1, & (x \in A), \\ 0, & (x \notin A). \end{cases}$$

Since a simple function $f(x)$ is a function, its range is fixed. Namely, the range of a simple function is the at most countable set in the space of extended real numbers $\overline{\boldsymbol{R}} = [-\infty,\ \infty]$.

Especially, we happen to say that a simple function is a step function if its range is the finite set in $\overline{\boldsymbol{R}}$.

Nevertheless, there are many ways of the expressions of a simple function $f(x)$ in the formula (1.1) because there are many varieties of the forms of the divisions Δ of E in the formula (1.2).

Thus, even if the range of a simple function $f(x)$ is fixed, we use the symbol $f_\Delta(x)$ in order to distinguish the simple functions whose expressions in the formula (1.1) are different.

Then we define the concept of the Lebesgue measurable functions in the following definition.

Definition 1.2 Let E be a measurable set in $\boldsymbol{R}^d$. Then we define that an extended real-valued function $f(x)$ defined on E is a **Lebesgue measurable function** if it satisfies the following conditions (i) and (ii):

(i) When we put $E(\infty) = \{x \in E;\ |f(x)| = \infty\}$, we have $E(\infty) \in \mathcal{M}$ and $\mu(E(\infty)) = 0$.

(ii) There exists a sequence of the simple functions $\{f_n(x);\ n \geq 1\}$ such that we have the limit

$$\lim_{n\to\infty} f_n(x) = f(x)$$

in the sense of pointwise convergence on $E \backslash E(\infty)$.

The condition (ii) of Definition 1.2 is equivalent to the following condition (iii):

(iii) For every point x in $E\backslash E(\infty)$ and an arbitrary positive number $\varepsilon > 0$, there exists a certain natural number n_0 such that we have the inequality

$$|f_n(x) - f(x)| < \varepsilon$$

for any natural number n such as $n \geq n_0$.

We say that a point x in $E(\infty)$ is a **singular point** of a function $f(x)$ in Definition 1.1 and Definition 1.2.

For simplicity, we say that a Lebesgue measurable function $f(x)$ is a **measurable function** or **measurable**

Example 1.1 Let a set E be a measurable set in $\boldsymbol{R}^d$. A simple function $f(x)$ and a continuous function $f(x)$ defined on E are measurable.

Theorem 1.1 *Let a set E be a measurable set in $\boldsymbol{R}^d$. Assume that two functions f and g are measurable on E. Then the following functions* (1) $\sim$ (10) *defined on E are also measurable*:

(1) $f + g$. (2) $f - g$. (3) fg.

(4) *f/g. Here we assume that $g(x) \neq 0$ holds on $x \in E$.*

(5) *αf. Here α is a real constant.* (6) *$|f|^p$. Here $p \neq 0$ is a real number.*

(7) $\sup(f,\ g)$. (8) $\inf(f,\ g)$. (9) $f^+ = \sup(f,\ 0)$.

(10) $f^- = -\inf(f,\ 0)$.

The functions $\sup(f,\ g)$ and $\inf(f,\ g)$ in Theorem 1.1 are defined in the following:

$$\sup(f,\ g)(x) = \sup(f(x),\ g(x)),\ (x \in E),$$
$$\inf(f,\ g)(x) = \inf(f(x),\ g(x)),\ (x \in E).$$

Further we have the relations

$$|f(x)| \geq f^+(x) \geq 0,\ |f(x)| \geq f^-(x) \geq 0.$$
$$f(x) = f^+(x) - f^-(x),\ |f(x)| = f^+(x) + f^-(x).$$

Theorem 1.2 *If a function $f(x)$ is measurable on E and we have the relations $F \subset E$ with $F \in \mathcal{M}$, the restriction $f_F(x) = f(x)|_F$ of $f(x)$ on F is measurable on F.*

Now we use the following notation. Let α and β be two arbitrary real numbers or $\pm\infty$. Then we put

$$E(f > \alpha) = \{x \in E;\ f(x) > \alpha\},$$
$$E(f \leq \alpha) = \{x \in E;\ f(x) \leq \alpha\},$$
$$E(f = \alpha) = \{x \in E;\ f(x) = \alpha\},$$
$$E(\alpha < f \leq \beta) = \{x \in E;\ \alpha < f(x) \leq \beta\},\ (\alpha < \beta).$$

Theorem 1.3 *Let $f(x)$ be a function defined on E. Then the following four propositions are equivalent*:

(1) *For an arbitrary real number α, we have $E(f > \alpha) \in \mathcal{M}_E$.*

(2) *For an arbitrary real number α, we have $E(f \leq \alpha) \in \mathcal{M}_E$.*

(3) *For an arbitrary real number α, we have $E(f \geq \alpha) \in \mathcal{M}_E$.*

(4) *For an arbitrary real number α, we have $E(f < \alpha) \in \mathcal{M}_E$.*

Corollary 1.1 *For a function $f(x)$ defined on E, the following* (1) *and* (2) *are equivalent*:

(1) *For an arbitrary real number* α, *we have* $E(f > \alpha) \in \mathcal{M}_E$.

(2) *For an arbitrary rational number* r, *we have* $E(f > r) \in \mathcal{M}_E$.

Corollary 1.2 *Assume that a function defined on* E *satisfies the conditions of Theorem* 1.3. *Then every set in the following* (1) $\sim$ (5) *belongs to* $\mathcal{M}_E$:

(1) $E(f = \alpha)$. *Here* α *is an arbitrary real number.*

(2) $E(f < \infty)$. (3) $E(f = \infty)$. (4) $E(f > -\infty)$.

(5) $E(f = -\infty)$.

Theorem 1.4 *For a function* $f(x)$ *defined on* E, *the following* (1) *and* (2) *are equivalent*:

(1) $f(x)$ *is measurable on* E. *Namely there exists a sequence of the simple functions* $\{f_n(x)\}$ *such that it converges to* $f(x)$ *in the sense of pointwise convergence on* $E \backslash E(\infty)$.

(2) *For an arbitrary real number* α, *we have* $E(f > \alpha) \in \mathcal{M}_E$.

If $f(x)$ is measurable, there exists a sequence of the simple functions $\{f_n(x)\}$ which converges to $f(x)$ in the sense of pointwise convergence on $E \backslash E(\infty)$ by virtue of the definition. Then it is the meaning of this theorem that the method of the concrete construction of one of such sequences of the simple functions is given.

We give the result in the following Corollary 1.3.

Corollary 1.3 *Assume that a function* $f(x)$ *is measurable on* E. *Then, for an arbitrary natural number* $n \geq 1$, *we put*

$$E_n^p = E\left(\frac{p}{n} \leq f < \frac{p+1}{n}\right), \ (p = 0, \ \pm 1, \ \pm 2, \ \cdots)$$

and we denote the defining function of E_n^p *as*

$$C_n^p(x) = \chi_{E_n^p}(x).$$

Then, if we define the simple function $f_n(x)$ *by the formula*

$$f_n(x) = \sum_{p=-\infty}^{\infty} \frac{p}{n} C_n^p(x), \ (x \in E),$$

the sequence of the simple functions $\{f_n(x)\}$ *converges to* $f(x)$ *in the sense of pointwise convergence on* $E \backslash E(\infty)$.

Theorem 1.5 *If a function $f(x)$ on E is measurable and $f(x) \geq 0$ holds for $x \in E$, there exists a sequence of the simple functions $\{f_n(x)\}$ which satisfies the conditions $f_n(x) \geq 0$, $(n \geq 1)$ and converges to $f(x)$ in the sense of pointwise convergence on $E \backslash E(\infty)$.*

Theorem 1.6 *If the functions $f_n(x)$, $(n \geq 1)$ defined on E are measurable, the following functions* $(1) \sim (5)$ *are also measurable on E:*

(1) $\sup_{n \geq 1} f_n(x)$. (2) $\inf_{n \geq 1} f_n(x)$. (3) $\overline{\lim}_{n \to \infty} f_n(x)$. (4) $\underline{\lim}_{n \to \infty} f_n(x)$.

(5) *If $f(x) = \lim_{n \to \infty} f_n(x)$ exists almost everywhere on E, then $f(x)$ is also measurable on E.*

If a certain property (P) concerning a function $f(x)$ or a sequence of measurable functions $\{f_n(x)\}$ holds everywhere on the set $E \backslash e$ for a certain null set e, then we say that this property (P) for the function $f(x)$ or the sequence of functions $\{f_n(x)\}$ holds **almost everywhere**.

For example, when we have the equality

$$f(x) = 0, \ (x \in E \backslash e, \ \mu(e) = 0),$$

we say that $f(x)$ is equal to 0 almost everywhere on E.

Further, when we have the limit

$$\lim_{n \to \infty} f_n(x) = f(x), \ (x \in E \backslash e, \ \mu(e) = 0),$$

we say that $f_n(x)$ converges to $f(x)$ almost everywhere on E.

We denote this as

$$\lim_{n \to \infty} f_n(x) = f(x), \ (\text{a.e. } x \in E).$$

Then the values of the limit function $f(x)$ happen to be undetermined on the null set e.

But we give one value to $f(x)$ and fix it at every point in such the null set e. We define the function as above in order to fix the definition of this function. Namely, if the domains are different for several functions, it is almost meaningless to state any proposition concerning such the functions.

In this case, even if we give what kind of value to $f(x)$ on a null set, this is an idea in order to express the proposition explicitly because this definition of $f(x)$ does not influence the value of the Lebesgue integral of $f(x)$.

Theorem 1.7 (Egorov's Theorem) *Assume that E is a measurable set in $\boldsymbol{R}^d$ such that $\mu(E) < \infty$ holds, and $f_n(x)$, $(n \geq 1)$ are measurable functions* *which have the finite value almost everywhere on* E*. Further, assume that there is the finite limit* $f(x) = \lim_{n\to\infty} f_n(x)$ *almost everywhere on* E*. Then, for an arbitrary positive number* $\varepsilon > 0$*, there exists a set* $F \in \mathcal{M}_E$ *such that we have the following* (1) *and* (2):

(1) *We have* $F \subset E$ and $\mu(E \backslash F) < \varepsilon$.

(2) $f_n(x)$ *converges to* $f(x)$ *uniformly on* F.

Corollary 1.4 *In the Theorem* 1.7*, we may have a closed set* F.

By virtue of Egorov's Theorem and Corollary 1.4, we have the following Theorem.

Theorem 1.8 (Lusin's Theorem) *Assume that* E *is a measurable set in* $\boldsymbol{R}^d$ *and* $f(x)$ *is a measurable function which has the finite value almost everywhere on* E*. Then, for an arbitrary positive number* $\varepsilon > 0$*, there exists a certain closed set* $F \subset E$ *such that we have the following* (1) *and* (2):

(1) *We have* $\mu(E \backslash F) < \varepsilon$.

(2) $f(x)$ *is continuous on* F.

2 Definition of the Lebesgue integral

In this section, we define the Lebesgue integral of a Lebesgue measurable function.

Assume that $d \geq 1$ holds and the d-dimensional Euclidean space $\boldsymbol{R}^d$ is the Lebesgue measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$.

Assume that a subset E of $\boldsymbol{R}^d$ is a Lebesgue measurable set.

Then, by restricting $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ on E, we have the d-dimensional Lebesgue measure space $(E,\ \mathcal{M},\ \mu)$ on E.

Here we define the Lebesgue integral of a Lebesgue measurable function $f(x)$ on E and denote this by the symbol

$$\int_E f(x)dx.$$

Although we denote a double integral or a triplet integral by the symbols

$$\int\int_E f(x)dx,\ \int\int\int_E f(x)dx$$

respectively, here we generally use the symbol of integral

$$\int_E f(x)dx.$$

Especially, we use the symbol of the double integral for the expression of Fubini's Theorem concerning the iterated integral.

In the following, we define the Lebesgue integral in two steps.

(1) The case where $f(x)$ is a simple function

In this case, we assume that a function $f(x)$ is defined in the formula

$$f(x) = \sum_{p=1}^{\infty} a_p \chi_{E_p}(x), \ (a_p \in \overline{\boldsymbol{R}}, \ p \geq 1) \tag{2.1}$$

for a division of E:

$$(\Delta): \ E = E_1 + E_2 + \cdots, \ (E_p \in \mathcal{M}_E, \ p \geq 1). \tag{2.2}$$

Then we define the Lebesgue integral of $f(x)$ as the sum of the series in the right hand side of the formula

$$\int_E f(x)dx = \sum_{p=1}^{\infty} a_p \mu(E_p). \tag{2.3}$$

We denote this as the symbol of the left hand side in the above. Here we assume that the series in the right hand side converges absolutely.

The sum of the absolutely convergent series in the right hand side of the formula (2.3) has the determined value independent of the choice of the expression of the function $f(x)$ in the formula (2.1).

Then we say that $f(x)$ is **Lebesgue integrable** on E. $f(x)$ is Lebesgue integrable on E if and only if $|f(x)|$ is Lebesgue integral on E.

This equivalence is understood because of the following consideration.

For the absolute function of the function $f(x)$ in the formula (2.1), we have the equality

$$|f(x)| = \sum_{p=1}^{\infty} |a_p| \chi_{E_p}(x). \tag{2.4}$$

Therefor we have the equality

$$\int_E |f(x)|dx = \sum_{p=1}^{\infty} |a_p| \mu(E_p). \tag{2.5}$$

Then if the series in the right hand side of the formula (2.3) converges absolutely if and only if the series in the right hand side of the formula (2.5) is convergent.

Remark 2.1 As for the convergence and the divergence of the series in the right hand side of the formula (2.3), we have the two cases of (1) convergence and (2) divergence.

In detail, we have two cases of (1-i) absolute convergence and (1-ii) conditional convergence for the case (1) and two cases of (2-i) divergence to either one of $\pm\infty$ and (2-ii) it vibrates and does not converge to any constant value in the case (2).

The case (1-i) is the definition of the Lebesgue integral and the case (1-ii) is the case where the integral converges conditionally.

Here, since we consider only the case where a simple function $f(x)$ is Lebesgue integrable, this means that we consider only the case (1-i) of Remark 2.1.

In general, as for the details of the convergence and divergence of the Lebesgue integral, we consider them in the method of calculation of the Lebesgue integral afterward.

(2) The case where $f(x)$ is a general measurable function

In this case, we assume that a function $f(x)$ is a general measurable function defined on E. Then we have a sequence of the simple functions $\{f_n(x)\}$ which converges to $f(x)$ on $E \backslash E(\infty)$ in the sense of pointwise convergence.

Here we assume that each $f_n(x)$ is Lebesgue integrable and there exists the limit

$$\lim_{n\to\infty} \int_E f_n(x)dx. \tag{2.6}$$

Then we say that this limit is the **Lebesgue integral** of $f(x)$ on E and denote it as

$$\int_E f(x)dx = \lim_{n\to\infty} \int_E f_n(x)dx. \tag{2.7}$$

Further we say that the Lebesgue integral (2.7) **converges absolutely** if the limit (2.6) does not depend on the choice of a sequence $\{f_n(x)\}$ of Lebesgue integrable simple functions which converges to $f(x)$ on $E \backslash E(\infty)$ in the sense of pointwise convergence and it is equal to the constant value.

Then we say that $f(x)$ is Lebesgue integrable on E. The usual Lebesgue integral is the Lebesgue integral in this case.

A function $f(x)$ defined on E is Lebesgue integrable if and only if the absolute function $|f(x)|$ is Lebesgue integrable.

Theorem 2.1 *When $f(x)$ is* Lebesgue *integrable on E, we choose the sequence of the simple functions $\{f_n(x)\}$ as in Corollary* 1.3. *Then the* Lebesgue *integral of $f(x)$ on E is given by the formula*

$$\int_E f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{p=-\infty}^{\infty} p\mu(E(\frac{p}{n} \leq f < \frac{p+1}{n})).$$

Theorem 2.2 *Assume that $f(x)$ is* Lebesgue *integrable on E. Now, if we define*

$$f^+(x) = \sup\{f(x),\ 0\},\ f^-(x) = -\inf\{f(x),\ 0\},\ (x \in E).$$

Then $f^+(x)$ and $f^-(x)$ are also Lebesgue *integrable on E and we have the equality*

$$\int_E f(x)dx = \int_E f^-(x)dx - \int_E f^-(x)dx.$$

Further we have the equality

$$\int_E |f(x)|dx = \int_E f^+(x)dx + \int_E f^-(x)dx.$$

Corollary 2.1 *Assume that $f(x)$ is* Lebesgue *integrable on E and $g(x)$ is* Lebesgue *measurable on E. Then, if we have the inequality $|g(x)| \leq |f(x)|$ on E, $g(x)$ is* Lebesgue *integrable on E.*

Further the Lebesgue integral (2.7) **converges conditionally** if the limit (2.6) has the various values depending on the choices of the sequences of Lebesgue integrable simple functions $\{f_n(x)\}$ which converge to $f(x)$ in the sense of pointwise convergence on $E \backslash E(\infty)$.

Until now, in this case, we have said that $f(x)$ is Lebesgue integrable in the extended sense on E and the Lebesgue integral in this case is the improper Lebesgue integral.

The Lebesgue integrable functions on E are the special case of the Lebesgue integrable functions in the extended sense.

If the limit (2.6) does not exist, we say that the Lebesgue integral **diverges**. In this case, the Lebesgue integral does not exist.

Remark 2.2 The case of the conditional convergence in Remark 2.1, (1-ii) means that the integral of the simple function is the improper Lebesgue integral.

In general, as for details of the situations of convergence or divergence of the Lebesgue integral, we study them in the section of the method of calculation of the Lebesgue integrals afterward.

3 Fundamental properties of the Lebesgue integrals

In this section, we study the fundamental properties of the Lebesgue integral.

Assume that a subset E of $\boldsymbol{R}^d$ is a Lebesgue measurable set and the d-dimensional Lebesgue measure space $(E,\ \mathcal{M},\ \mu)$ is defined on E. Here we assume $d \geq 1$.

3.1 The fundamental properties of the Lebesgue integral

In this paragraph, we study the fundamental properties of the Lebesgue integral.

As for the all formulas of all theorems in this paragraph, we can easily prove that these formulas are true for the Lebesgue integrable simple functions. For the general Lebesgue measurable functions, we prove these formulas by taking limits from all formulas for the Lebesgue measurable simple functions by virtue of the definition of the Lebesgue integral. Therefore we omit the details of the proofs here.

Theorem 3.1.1 *Assume that a function $f(x)$ is Lebesgue integrable on E and F is a measurable subset of E. Then the restriction $f_F(x) = f(x)|_F$ of $f(x)$ on F is Lebesgue integrable on F and we have the equality*

$$\int_E f_F(x)dx = \int_F f(x)dx.$$

Namely the function $f(x)$ is Lebesgue integrable on F.

Theorem 3.1.2 *Assume that a set E is a Lebesgue measurable set in $\boldsymbol{R}^d$ and a function $f(x)$ is integrable on E. If $E = E_1 + E_2$ is a division of E and E_1* and *E_2 are Lebesgue measurable, we have the equality*

$$\int_E f(x)dx = \int_{E_1} f(x)dx + \int_{E_2} f(x)dx.$$

Theorem 3.1.3 *If a function $f(x)$ is* Lebesgue *integrable on E, we have the inequality*

$$|\int_E f(x)dx| \leq \int_E |f(x)|dx.$$

Theorem 3.1.4 *Assume that a set E is* Lebesgue *measurable in $\boldsymbol{R}^d$ and two functions $f(x)$ and $g(x)$ are* Lebesgue *integrable on E. Then we have the following* (1) $\sim$ (4):

(1) *$f(x)+g(x)$ is also* Lebesgue *integrable on E and we have the equality*

$$\int_E \{f(x)+g(x)\}dx = \int_E f(x)dx + \int_E g(x)dx.$$

(2) *For an arbitrary real constant α, $\alpha f(x)$* is *also* Lebesgue *integrable on E and we have the equality*

$$\int_E \{\alpha f(x)\}dx = \alpha \int_E f(x)dx.$$

(3) *If $f(x) \geq 0$ holds on E, we have the inequality*

$$\int_E f(x)dx \geq 0.$$

(4) *If $f(x) \geq g(x)$ holds on E, we have the inequality*

$$\int_E f(x)dx \geq \int_E g(x)dx.$$

Corollary 3.1.1 *If two functions $f(x)$ and $g(x)$ are* Lebesgue *integrable on E, then, for two arbitrary real constants α and β, $\alpha f(x)+\beta g(x)$ is also* Lebesgue *integrable on E and we have the equality*

$$\int_E \{\alpha f(x)+\beta g(x)\} = \alpha \int_E f(x)dx + \beta \int_E g(x)dx.$$

Theorem 3.1.5 *If a function $f(x)$ is* Lebesgue *integrable on E, we have the following* (1) *and* (2):

(1) *If $\mu(E)=0$ holds, we have*

$$\int_E f(x)dx = 0.$$

(2) *We have $\mu(E(f=\infty)) = \mu(E(f=-\infty)) = 0$.*

Corollary 3.1.2 *Assume that two functions $f(x)$ and $g(x)$ are* Lebesgue *measurable on E and $f(x) = g(x)$ holds almost everywhere on E. Then, if $f(x)$ is* Lebesgue *integrable on E, $g(x)$ is also* Lebesgue *integrable on E and we have the equality*

$$\int_E f(x)dx = \int_E g(x)dx.$$

By virtue of Corollary 3.1.2, if two Lebesgue integrable functions are equal almost everywhere on E, we have not to distinguish their Lebesgue integrals.

Theorem 3.1.6 *If a function $f(x)$ is* Lebesgue *integrable on E, $E(f \neq 0)$ is equal to a union of at most countable sets with the finite* Lebesgue *measures.*

Theorem 3.1.7(The first mean value theorem) *Assume that a set E is measurable and a function $f(x)$ is a bounded measurable function on E and $g(x)$ is* Lebesgue *integrable on E.*

Then, if we put

$$m = \inf_{x \in E} f(x), \; M = \sup_{x \in E} f(x),$$

we have the following (1) *and* (2):

(1) *$f(x)g(x)$ is* Lebesgue *integrable on E.*

(2) *There exists a real constant α with $m \leq \alpha \leq M$ such that we have the equality*

$$\int_E f(x)|g(x)|dx = \alpha \int_E |g(x)|dx.$$

Corollary 3.1.3 *Assume that a function $f(x)$ is continuous on a bounded closed domain and $g(x)$ is* Lebesgue *integrable on E and $g(x) \geq 0$ holds for $x \in E$. Then there exists a certain point $x_0 \in E$ such that we have the equality*

$$\int_E f(x)g(x)dx = f(x_0) \int_E g(x)dx.$$

Theorem 3.1.8 *Assume that E is a* Lebesgue *measurable set of $\boldsymbol{R}^d$ and a function $f(x)$ is* Lebesgue *integrable on E. Then, for an arbitrary positive number $\varepsilon > 0$, there exists a continuous function $f_\varepsilon(x)$ which is identically zero outside a certain bounded measurable set such that we have the inequalities*

$$\left| \int_E f(x)dx - \int_E f_\varepsilon(x)dx \right| \leq \int_E \mid f(x) - f_\varepsilon(x) \mid dx < \varepsilon.$$

Theorem 3.1.9 *If a function $f(x)$ is* Lebesgue *integrable on $\boldsymbol{R}^d$, then, for an arbitrary $y \in \boldsymbol{R}^d$, the function $f(x+y)$ is* Lebesgue *integrable as a function of x and the function $f(-x)$ is also* Lebesgue *integrable as a function of x. Then we have the equalities*

$$\int_{\boldsymbol{R}^d} f(x+y)dx = \int_{\boldsymbol{R}^d} f(-x)dx = \int_{\boldsymbol{R}^d} f(x)dx.$$

Theorem 3.1.10 *If a function $f(x)$ is* Lebesgue *integrable on $\boldsymbol{R}^d$, we have the equality*

$$\lim_{y \to 0} \int_{\boldsymbol{R}^d} \mid f(x+y) - f(x) \mid dx = 0.$$

3.2 Lebesgue integral and limit

In this paragraph, we study the relations of the d-dimensional Lebesgue integral and limit.

Theorem 3.2.1 *Assume that E is a* Lebesgue *measurable set of $\boldsymbol{R}^d$ and we have the division*

$$E = E_1 + E_2 + \cdots$$

by using the sequence $\{E_n;\ n \geq 1\}$ of mutually disjoint Lebesgue *measurable sets. Then, if a function $f(x)$ is* Lebesgue *integrable on E, we have the equality*

$$\int_E f(x)dx = \int_{E_1} f(x)dx + \int_{E_2} f(x)dx + \cdots .$$

Further, if a function $f(x)$ is Lebesgue *measurable on every set E_n, $(n \geq 1)$ and the condition*

$$\sum_{n=1}^{\infty} \int_{E_n} \mid f(x) \mid dx < \infty$$

is satisfied, we also have the equality in the above.

Corollary 3.2.1 *Assume that E is a* Lebesgue *measurable set of $\boldsymbol{R}^d$ and $\{E_n;\ n \geq 1\}$ is a monotone increasing sequence of* Lebesgue *measurable sets and satisfies the condition*

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Then, if a function $f(x)$ is Lebesgue *integrable on E, then, for an arbitrary positive number $\varepsilon > 0$, there exists a certain natural number n_0 such that, for $n \geq n_0$, we have the estimate*

$$\int_{E \setminus E_n} |f(x)| dx < \varepsilon.$$

Especially, we have the equality

$$\lim_{n \to \infty} \int_{E_n} f(x) dx = \int_E f(x) dx.$$

Remark 3.2.1 When the Lebesgue integral

$$\int_E f(x) dx$$

of a function $f(x)$ converges conditionally, we have the limit in Corollary 3.2.1 for a special choice of the sequence $\{E_n\}$ of the Lebesgue integrable sets in Corollary 3.2.1 in the above.

Corollary 3.2.2 *Assume that E is a* Lebesgue *measurable set of $\boldsymbol{R}^d$ and a function $f(x)$ is* Lebesgue *integrable on E. Now, we put*

$$E_n = E(|f| < n), \ (n \geq 1).$$

Then, for an arbitrary positive number $\varepsilon > 0$, there exists a certain natural number n_0 such that, for $n \geq n_0$, we have the estimate

$$\int_{E \setminus E_n} |f(x)| dx < \varepsilon.$$

Especially, we have the equality

$$\lim_{n \to \infty} \int_{E_n} f(x) dx = \int_E f(x) dx.$$

The following Theorem 3.2.2 means the absolute continuity of the indefinite integral.

Theorem 3.2.2 *Assume that E is a* Lebesgue *measurable set of $\boldsymbol{R}^d$ and a functions $f(x)$ is* Lebesgue *integrable on E. Then, for arbitrary positive number $\varepsilon > 0$, there exists a certain positive number $\delta > 0$ such that we have the estimate*

$$|\int_e f(x) dx| < \varepsilon$$

if we have $\mu(e) < \delta$ *for any* Lebesgue *measurable subset* $e \subset E$.

Theorem 3.2.3(Lebesgue's bounded convergence theorem) *Assume that* E *is a bounded measurable set in* $\boldsymbol{R}^d$. *If a sequence* $\{f_n(x); n \geq 1\}$ *of the uniformly bounded* Lebesgue *measurable functions converges to* $f(x)$ *almost everywhere on* E, *we have the equality*

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

Theorem 3.2.4(Lebesgue's convergence theorem) *Assume that* E *is a measurable set in* $\boldsymbol{R}^d$. *Then assume that a sequence* $\{f_n(x);\ n \geq 1\}$ *of* Lebesgue *measurable functions converges to the finite limit* $f(x)$ *almost everywhere on* E. *Further, if there exists a Lebesgue integrable function* $\Phi(x)$, (≥ 0) *on* E *such that we have the estimates*

$$|f_n(x)| \leq \Phi(x),\ (x \in E,\ n \geq 1),$$

we have the equality

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

We obtain the theorem of the termwise integration by virtue of the Lebesgue's convergence theorem.

Theorem 3.2.5(Theorem of the termwise convergence) *Assume that* E *is a* Lebesgue *measurable set in* $\boldsymbol{R}^d$ *and* $\{f_n(x);\ n \geq 1\}$ *is a sequence of the* Lebesgue *measurable functions on* E. *Now we put*

$$f(x) = f_1(x) + f_2(x) + \cdots .$$

Then, if the series in the right hand side of the formula in the above converges almost everywhere on E *and there exists a* Lebesgue *integrable function* $\Phi(x)$, (≥ 0) *on* E *such that we have the estimates*

$$\left| \sum_{p=1}^{n} f_p(x) \right| \leq \Phi(x),\ (x \in E)$$

for an arbitrary $n \geq 1$, *we can obtain the theorem of the termwise integration. Namely we have the equality*

$$\int_E f(x)dx = \int_E f_1(x)dx + \int_E f_2(x)dx + \cdots .$$

Corollary 3.2.3 *Assume that* E, $\{f_n(x)\}$ *and* $f(x)$ *are the same as in Theorem* 3.2.5. *Then we assume that the following condition* (i) *or* (ii) *is satisfied*:

(i) *There exists a* Lebesgue *integrable function* $\Phi(x)$, (≥ 0) *on* E *such that we have the estimates*

$$\sum_{p=1}^{n} |f_p(x)| \leq \Phi(x),\ (x \in E,\ n \geq 1).$$

(ii) *We have the condition*

$$\sum_{p=1}^{\infty} \int_E |f_p(x)|dx < \infty.$$

Then we have the theorem of the termwise integration.

Theorem 3.2.6(Beppo Levi's theorem) *Assume that* E *is a* Lebesgue *measurable set in* $\boldsymbol{R}^d$ *and* $\{f_n(x);\ n \geq 1\}$ *is a monotone increasing sequence of the* Lebesgue *integrable functions on* E*. Further, assume that the monotone increasing sequence*

$$\{ \int_E f_n(x)dx \}$$

is bounded above. Then, if we put

$$\lim_{n\to\infty} f_n(x) = f(x),\ (x \in E),$$

the function $f(x)$ *has the finite value almost everywhere on* E*, and it is* Lebesgue *integrable on* E*, and we have the equality*

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

Here we give the fact used in the proof of the theorem in the above in the following Corollary.

Corollary 3.2.4 *Assume that* E *is a* Lebesgue *measurable set in* $\boldsymbol{R}^d$ *and* $\{E_n;\ n \geq 1\}$ *is a monotone increasing sequence of* Lebesgue *measurable sets in* $\boldsymbol{R}^d$ *so that we have the equality*

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Further, if a Lebesgue *measurable function on* E *is* Lebesgue *integrable on each* E_n *and we have the condition*

$$\lim_{n\to\infty} \int_{E_n} |f(x)|dx < \infty,$$

then $f(x)$ *is* Lebesgue *integrable on* E *and we have the equality*

$$\lim_{n\to\infty} \int_{E_n} f(x)dx = \int_E f(x)dx.$$

Corollary 3.2.5 *Assume that* E *is a* Lebesgue *measurable set in* $\boldsymbol{R}^d$ *and* $\{f_n(x);\ n \geq 1\}$ *is a monotone increasing sequence of the* Lebesgue *integrable functions on* E. *Then, if the limit*

$$\lim_{n\to\infty} f_n(x) = f(x),\ (x \in E)$$

has the finite value almost everywhere on E *and it is* Lebesgue *integrable on* E, *we have the equality*

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

Next we prove the Fatou's Lemma as the corollary of Beppo Levi's theorem.

At first, we remark that the Fatou's Lemma is used many times in the following form. Assume that we have the equality

$$\lim_{n\to\infty} f_n(x) = f(x),\ (x \in E)$$

for the Lebesgue integrable functions $f_n(x),\ (n \geq 1)$ on a measurable set E. If we have the condition

$$\varliminf_{n\to\infty} \int_E f_n(x)dx < \infty,$$

then $f(x)$ is also Lebesgue integrable on E and we have the inequality

$$\int_E f(x)dx \leq \varliminf_{n\to\infty} \int_E f_n(x)dx.$$

Here we prove the Fatou's Lemma which is fairly more generalized.

Theorem 3.2.7(Fatou's Lemma) *Assume that* E *is a* Lebesgue *measurable set in* $\boldsymbol{R}^d$ *and* $\{f_n(x);\ n \geq 1\}$ *is a sequence of the* Lebesgue *integrable nonnegative functions on* E *such that we have the condition*

$$\varliminf_{n\to\infty} \int_E f_n(x)dx < \infty.$$

Then the inferior limit

$$f(x) = \varliminf_{n\to\infty} f_n(x)$$

is Lebesgue *integrable on E and we have the inequality*

$$\int_E f(x)dx = \int_E (\varliminf_{n\to\infty} f_n(x))dx \leq \varliminf_{n\to\infty} \int_E f_n(x)dx.$$

The following theorem 3.2.8 is the result concerning the differentiation under the integral symbol.

Theorem 3.2.8 *Assume that E is a* Lebesgue *measurable set in $\boldsymbol{R}^d$ and $(a,\ b)$ is an interval in $\boldsymbol{R}$. Assume that a function $f(x,\ t)$ is defined on the set*

$$E \times (a,\ b) = \{(x,\ t);\ x \in E,\ t \in (a,\ b)\}$$

and it satisfies the following conditions (i) $\sim$ (iii):

(i) *For an arbitrary chosen and fixed $t \in (a,\ b)$, $f(x,\ t)$ is* Lebesgue *integrable on E.*

(ii) *For an almost every $x \in E$, $f(x,\ t)$ is differentiable with respect to t. Then we denote its partial derivative with respect to t as $f_t(x,\ t)$.*

(iii) *There exists a* Lebesgue *integrable function $\Phi(x)$, (≥ 0) on E such that we have the estimate*

$$|f_t(x,\ t)| \leq \Phi(x),\ (x \in E,\ t \in (a,\ b)).$$

Then, if we put

$$F(t) = \int_E f(x,\ t)dx,$$

$F(t)$ is differentiable on $(a,\ b)$ and we have the equality

$$F'(t) = \int_E f_t(x,\ t)dx.$$

4 Calculation of the d-dimensional Lebesgue integral

In this section, we study the calculation of the d-dimensional Lebesgue integral by way of approximating the integral domain by the approximating direct family of the bounded closed sets.

Assume that the integral domain E is a Lebesgue measurable set in $\boldsymbol{R}^d$ and the integrand $f(x)$ is Lebesgue measurable on E. Assuming that A is a direct set, we consider the direct family $\{E_\alpha;\ \alpha \in A\}$ of the bounded closed sets in E.

Now, we say that a direct family $\{E_\alpha\}$ **converges** to E if, for an arbitrary bounded closed set K included in E, there exists a certain $\alpha_0 \in A$ such that, for an arbitrary α with $\alpha \geq \alpha_0,\ K \subset E_\alpha$ holds. Then the direct family $\{E_\alpha\}$ is said to be an **approximating direct family**.

Especially, if, for $A = \{1,\ 2,\ 3,\ \cdots\}$, $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$ holds, the sequence $\{E_n\}$ **converges** to E **monotonely**. In general, when a sequence $\{E_n\}$ converges to E, the sequence $\{H_n\}$ converges to E monotonely for $H_n = E_1 \cup E_2 \cup \cdots \cup E_n,\ (n \geq 1)$.

Assume that the set $E(\infty)$ of all singular points of $f(x)$ has the measure 0. Then $E \backslash E(\infty)$ is also a Lebesgue measurable set.

Further, assume that $f(x)$ is Lebesgue integrable on an arbitrary bounded closed set included in $E \backslash E(\infty)$. In order to be so, any bounded closed set included in $E \backslash E(\infty)$ and the set $E(\infty)$ of all singular points of $f(x)$ do not contact. Namely these sets has a positive distance away from each other. This is the reason why we construct an approximating direct family $\{E_\alpha,\ \alpha \in A\}$ by using the bounded closed sets E_α.

Now assume that a direct family $\{E_\alpha\}$ of the bounded closed sets which converges to $E \backslash E(\infty)$ is the **approximating direct family** of $E \backslash E(\infty)$.

Then, if, for an approximating direct family $\{E_\alpha\}$ of $E \backslash E(\infty)$,

$$I(E_\alpha) = \int_{E_\alpha} f(x)dx$$

converges in the sense of Moore-Smith limit, the limit

$$I = \lim_{\alpha} I(E_\alpha)$$

is equal to the **Lebesgue integral** of $f(x)$ on E

$$I = \int_E f(x)dx.$$

Here we say that this Lebesgue integral **converges absolutely** if the value I of this Lebesgue integral does not depend on the choice of the approximating direct family $\{E_\alpha\}$ of $E \backslash E(\infty)$.

Further we say that this Lebesgue integral **converges conditionally** if the value I depends on the choice of the approximating direct family $\{E_\alpha\}$ of $E \backslash E(\infty)$.

Then we say that the Lebesgue integral exists if the Lebesgue integral converges absolutely or conditionally. We say that $f(x)$ is **integrable** if the Lebesgue integral of $f(x)$ converges absolutely.

We say that the Lebesgue integral **diverges** if the Lebesgue integral does not exist.

A function $f(x)$ is Lebesgue integrable on E if and only if $|f(x)|$ is Lebesgue integrable on E.

By the consideration in the above, so to say, the improper Lebesgue integral is cleared to be the calculation of the Lebesgue integral by approximating the integral domain using the approximating direct family of the bounded closed sets. Until now, we considered that the Lebesgue integral is the integral which converges absolutely. We have said this as the Lebesgue integral in the narrow sense. On the other hand, we have said that this is the improper Lebesgue integral if the Lebesgue integral converges including the case where the Lebesgue integral converges conditionally.

In this paper, we remark that the Lebesgue integral defined in section 2 is the unified definition of the Lebesgue integral including the Lebesgue integral in the narrow sense and the improper Lebesgue integral.

Remark 4.1 Assume that E is a measurable set in $\boldsymbol{R}^d$ and $f(x)$ is an extended real-valued measurable function defined on E.

Then there exist a direct family $\{f_\Delta(x)\}$ of the simple functions which converges to $f(x)$ on $E\backslash E(\infty)$ in the sense of pointwise convergence and an approximating direct family $\{E_\alpha\}$ of $E\backslash E(\infty)$ composed of the bounded closed subsets. Thereby we have the limits (I), (II) in the sense of Moore-Smith convergence as follows:

(I) $$\int_E f(x)dx = \lim_\Delta \int_E f_\Delta(x)dx.$$

(II) $$\int_E f(x)dx = \lim_\alpha \int_{E_\alpha} f(x)dx.$$

In the case (I), we remark that we may use the sequence $\{f_n(x)\}$ of the simple functions which converges to $f(x)$ on $E\backslash E(\infty)$ in the sense of pointwise convergence.

Then the convergence or the divergence of the Lebesgue integrals in (I) and (II) are equivalent.

Further, in the case of convergence, the absolute convergence or the conditional convergence of the Lebesgue integrals in (I) and (II) are equivalent.

The Lebesgue integral in (I) is calculated by using the approximation of the function $f(x)$ by the direct family of the simple functions. The Lebesgue integral in (II) is calculated by using the approximating direct family of $E\backslash E(\infty)$ composed of the bounded closed subsets.

Thus, in order to calculate the Lebesgue integral, there exists the either one of the methods of calculations such as the calculation by using the approximation of the function or the calculation by using the approximation of the integral domain.

Then we have the following relation in the table 4.1 concerning the convergence or the divergence of the Lebesgue integral of $f(x)$.

Table 4.1 Convergence and divergence of the Lebesgue integral

$$\left(\begin{array}{l}\text{conv.=convergence, div.=divergence,}\\ \text{abs.conv.=absolute convergence,}\\ \text{cond.conv.=conditional convergence.}\end{array}\right)$$

$\int_E f(x)dx$	$\int_E \lvert f(x)\rvert dx$	$\int_E f^+(x)dx$	$\int_E f^-(x)dx$
abs.conv.	conv.	conv.	conv.
div.	div.	conv.	div.
div.	div.	div.	conv.
cond.conv. or div.	div.	div.	div.

Remark 4.2 In the case where the Lebesgue integral converges absolutely in Table 4.1, the value of this Lebesgue integral is determined independently to the choice of the approximating direct family $\{E_\alpha;\ \alpha \in A\}$ of E.

Only in this case of the absolute convergence, the Lebesgue integral has a determined meaning.

In Table 4.1, the cases where the Lebesgue integral $\displaystyle\int_E f(x)dx$ diverges to $\pm\infty$ are the following cases (1) and (2):

(1) $\displaystyle\int_E f^+(x)dx < \infty,\ \int_E f^-(x)dx = \infty.$

(2) $\displaystyle\int_E f^+(x)dx = \infty,\ \int_E f^-(x)dx < \infty.$

In these cases, the Lebesgue integrals do not exist.

Nevertheless, in this case, the set function $m(A)$ on $\mathcal{M}_E$ is defined by the formula

$$m(A) = \int_A f(x)dx$$

for a Lebesgue measurable set A in E.

Here $\mathcal{M}_E$ is the family of all Lebesgue measurable sets in E. Thereby the Lebesgue-Stieltjes measure space $(E,\ \mathcal{M}_E,\ m)$ on E is defined. In the case (1), the total measure is equal to $m(E) = -\infty$, and in the case (2), the total measure is equal to $m(E) = \infty$. This measure space has the determined meaning as a σ-finite measure space

Then the Lebesgue integral of $f(x)$ on E itself does not exist. But the indefinite integral of $f(x)$ on a Lebesgue measurable set A in E is defined by the formula

$$m(A) = \int_A f(x)dx$$

and its value is determined as a finite real value or $-\infty$ or ∞. On the other hand, in the case where the Lebesgue integral converges conditionally or diverges in Table 4.1, the Lebesgue integral converges or diverges according to the choice of the approximating direct family $\{E_\alpha;\ \alpha \in A\}$ of E.

Then, in the case where the Lebesgue integral diverges, we cannot give any meaning to this integral.

Nevertheless, in the case where the Lebesgue integral converges conditionally, we can define its value as the value which is meaningful mathematically.

But, in this case, it is hard to make the general theory and we have to design a way to give its meaning according to each function with the singular points.

Theorem 4.1 *Assume that E is a measurable set in $\boldsymbol{R}^d$ and a function $f(x)$ is an extended real-valued measurable function which is nonnegative. Then, if, for an approximating direct family $\{E_\alpha\}$ of $E\backslash E(\infty)$ composed of the bounded closed sets, there exists a* Moore-Smith *limit*

$$\lim_\alpha I(E_\alpha) = \lim_\alpha \int_{E_\alpha} f(x)dx,$$

the Lebesgue *integral of $f(x)$ on E converges absolutely.*

Theorem 4.2 *Assume that E and $f(x)$ are as same as in Theorem* 4.1. *Then the* Lebesgue *integral $\displaystyle\int_E f(x)dx$ converges if and only if, for every bounded closed set H in $E\backslash E(\infty)$,*

$$I(H) = \int_H f(x)dx$$

is bounded.

Theorem 4.3 *Assume that E is a bounded measurable set in $\boldsymbol{R}^d$ and a function $f(x)$ is* Lebesgue *integrable on E. Further assume that $f(x) \geq 0$ holds. Assume that a sequence $\{E_n\}$ of the measurable subsets of E satisfies the following conditions* (i) *and* (ii):

(i) *$f(x)$ is integrable on E_n, $(n \geq 1)$.*

(ii) *$\mu(E\backslash E_n) \to 0$, $(n \to \infty)$ holds.*

Then we have the equality

$$\lim_{n\to\infty}\int_{E_n} f(x)dx = \int_E f(x)dx.$$

Remark 4.3 If there does not exist the integral of a function $f(x)$ on E in Theorem 4.3, we have $\displaystyle\int_E f(x)dx = \infty$. Then, if the other conditions are the same as in the Theorem, we have the equality

$$\lim_{n\to\infty}\int_{E_n} f(x)dx = \infty = \int_E f(x)dx.$$

Remark 4.4 Assume that a subset E of $\boldsymbol{R}^d$ is Jordan measurable or Lebesgue measurable and a function $f(x)$ on E is Jordan measurable or Lebesgue measurable. Now we classify these functions in the following four classes:

(R) The set of all Riemann integrable functions.

(ER) The set of all functions whose Riemann integrals exist.

(L) The set of all Lebesgue integrable functions.

(EL) The set of all functions whose Lebesgue integrals exist.

Then we have the following inclusion relations

$$(\mathrm{R}) \subset (\mathrm{ER}) \cap (\mathrm{L}) \subset (\mathrm{L}) \subset (\mathrm{ER}) \cup (\mathrm{L}) \subset (\mathrm{EL}).$$

References

[1] Yoshifumi Ito, *Analysis*, Vol. I, Science House, 1991, (in Japanese).

[2] ——— , *Analysis*, Vol. II, Science House, 1998, (in Japanese), (Out of print).

[3] ——— , *Axioms of Arithmetic*, Science House, 1999, (in Japanese).

[4] ——— , *Method of Renormalization and Distributions*, J. Math. Univ. Tokushima, **35** (2001), 35-55.

[5] ——— , *Foundation of Analysis*, Science House, 2002, (in Japanese).

[6] ——— , *Theory of Measure and Integration*, Science House, 2002, (in Japanese).

[7] ——— , *Analysis*, Vol. II, (rev. ed.), Science House, 2002, (in Japanese).

[8] ——— , *Why the area is obtained by the integration*, Mathematics Seminar, **44, no.6** (2005), pp.50-53, (in Japanese).

[9] ——— , *New Meanings of Conditional Convergence of the Integrals*, Real Analysis Symposium 2007, Osaka, pp.41-44, (in Japanese).

[10] ——— , *Definition and Existence Theorem of Jordan Measure*, Real Analysis Symposium 2010, Kitakyushu, pp.1-4, 2011.

[11] ——— , *Differential and Integral Calculus* II, − *Theory of Riemann Integral* −, preprint, 2010, (in Japanese).

[12] ——— , *Theory of Lebesgue Integral*, preprint, 2010, (in Japanese).

[13] ——— , *Differential and Integral Calculus* I, − *Theory of Differentiation* −, preprint, 2011, (in Japanese).

[14] ——— , RS-*integral and* LS-*integral*, preprint, 2011, (in Japanese).

[15] ——— , *Line Integral and Surface Integral*, preprint, 2011, (in Japanese).

[16] ——— , *Introduction to Analysis*, preprint, 2014, (in Japanese).

[17] ——— , *Axiomatic Method of Measure and Integration* (I). *Definition and Existence Theorem of the Jordan Measure*, J. Math. Tokushima Univ., **52** (2018), 1-15.

[18] ——— , *Axiomatic Method of Measure and Integration* (II). *Definition of the Riemann Integral and its Fundamental Properties*, J. Math.Tokushima Univ., **52** (2018), 17-38.

[19] ——— , *Axiomatic Method of Measure and Integration* (III). *Definition and Existence Theorem of the Lebesgue Measure*, preprint, 2019, (in Japanese).