J. Math. Tokushima Univ.
Vol. 54 (2020), 1 − 12

On Polygonal Square Triangular Numbers

By

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Received September 18 2020

Abstract

A pentagonal square triangular number is a number which is a pentagonal number $P_5(\ell)$, a square y^2 and a triangular number $P_3(m)$ at the same time. It would be well known for the specialists that there exists no pentagonal square triangular number except for $P_3(1) = 1^2 = P_5(1) = 1$. But we don't know any simple reference of the proof of this fact in print. The object of this note is to provide a such reference. Here we shall present three independent proofs of this fact one of which was already referred in the net article [24].

2010 Mathematics Subject Classification. Primary 11D09; Secondary 11G05

1 Introduction

Determination of the pentagonal square triangular number is proposed as the problem 29.4 (c) of Silverman's text "Friendly Introduction to Number Theory". Hence J. Silverman probably knew the answer and the proof of this fact. On the other hand, E. Deza declares that the problem is still an open problem in the text "Figurate Numbers". The following explanation was written in the net article "Pentagonal Square Triangular Number" of Wolfram Mathworld. "It is almost certain that no other solution exists except for 1, although no proof of this fact appears to have yet appeared in print." In the article, they continued that the problem was finally settled by J. Sillcox in 2003 and 2006 as follows. It is easy to verify that the determination of pentagonal square triangular number is equivalent to solve the simultaneous pell equations $x^2-2y^2=1, z^2-6y^2=1$ with even y. Sillcox has pointed out that this problem is the special case $R=2, S=6$ of W. S. Anglin' s results which determined all the positive integer solutions of simultaneous pell equations $x^2-Ry^2=1, z^2-Sy^2=1$ for all the cases $0<R<S\leq 200$ [2] (1996).

In the following section, we shall explain the proof the above fact of W. S. Anglin and also give two other proofs. We note that this paper is the growth of the bachelor's thesis of the second and the third authors.

2 Pentagonal Square Triangular Number

2.1 Baker theory on linear forms in logarithms

Let $P_k(n)$ be the $n-$th $k-$gonal number, i.e., the number of dots of a regular $k-$gon with n dots on each side. Then $P_k(n)$ is written as

$$P_k(n)=\frac{n((k-2)n-(k-4))}{2}.$$

In the following the set of all the $k-$gonal numbers $\{P_k(n)\mid n\in\mathbb{N}\}$ will be denoted by P_k. Assume

$$P_3(m)=\frac{m(m+1)}{2}=y^2=P_5(n)=\frac{n(3n-1)}{2}.$$

Putting $X=6n-1$, $Y=2m+1$ and $Z=2y$, we obtain the following simultaneous pell equations

$$\begin{cases} X^2-6Z^2 &= 1,\\ Y^2-2Z^2 &= 1.\end{cases}$$

Then, in the article of the pentagonal square triangular number of Wolfram Mathworld, J. Sillcox referred the result of W. S. Anglin [2] (1996). Actually

Anglin determined all the positive integer solutions of the following simultaneous pell equations

$$\begin{cases} X^2 - SZ^2 &= 1, \\ Y^2 - RZ^2 &= 1. \end{cases}$$

where $0 < R < S \leq 200$ and not square. Hence our case is $R = 2, S = 6$ and J. Sillcox pointed out that the solution of the above simultaneous pell equation is only the trivial one $P_3(1) = 1^2 = P_5(1)$ in 2003 and 2006, although no proof of this fact appears and to have yet appeared in print. Here we quote that the proof of Anglin is based on Baker's theory on the linear forms in logarithms. The methods are described in Anglin's paper [2], or in section 4.6 of Anglin's text [1], or in the section 1 of the first author's paper [12].

Remark 2.1 *We note one can determine the solutions of simultaneous pell equations by several other methods as in* [12]. *Here we shall explain two other methods in the following sections.*

2.2 Square Terms in Binary Recurrence Sequences

$$P_3(m) = \frac{m(m+1)}{2} = y^2 = P_5(n) = \frac{n(3n-1)}{2}.$$

implies

$$\begin{cases} X^2 - 6Z^2 &= 1, \\ Y^2 - 2Z^2 &= 1, \end{cases} \Longleftrightarrow \begin{cases} X^2 - 3Y^2 &= -2, \\ Y^2 - 2Z^2 &= 1, \end{cases}$$

where $X = 6n-1$, $Y = 2m+1$ and $Z = 2y$. Let us define the binary recurrence sequences a_k, b_k by putting $a_{k+1} = 4a_k - a_{k-1}, b_{k+1} = 4b_k - b_{k-1}$, with initial terms $a_0 = 1, a_1 = 2, b_0 = 0, b_1 = 1$. Then $\{(a_k, b_k) \mid k \geq 0\}$ is the set of all the non negative integer solutions of the pell equation $x^2 - 3y^2 = 1$. Since 2 totally ramifies in the real quadratic field $\mathbb{Q}(\sqrt{3})$, we know $X + Y\sqrt{3} = (1+\sqrt{3})(a_k + b_k\sqrt{3})$, i.e.,

$$X = a_k + 3b_k, \text{ and } Y = a_k + b_k.$$

Moreover $a_{k+1} + b_{k+1}\sqrt{3} = (a_k + b_k\sqrt{3})(2+\sqrt{3})$ implies two relations

$$a_{k+1} = 2a_k + 3b_k \text{ and } b_{k+1} = a_k + 2b_k.$$

Hence we have

$$Y^2 - 1 = (a_k + b_k)^2 - 1 = a_k^2 - 1 + b_k^2 + 2a_kb_k = 4b_k^2 + 2a_kb_k = 2b_kb_{k+1}.$$

Combining the facts $Z^2 = b_kb_{k+1}$ and $(b_k, b_{k+1}) = 1$, we know $b_k = \square$ and $b_{k+1} = \square$. Now let us recall Ljunggren's classical result of [14].

Theorem 2.2 (Ljunggren) *The equation $x^2 - Dy^4 = 1$, where D is positive and not a perfect square has at most two positive integer solutions. If there are two solutions, these solutions are given by either $x + y\sqrt{D} = \varepsilon_D, \varepsilon_D^2$ or by $x+y\sqrt{D} = \varepsilon_D, \varepsilon_D^4$. Here ε_D (> 1) is the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{D})$. Moreover the exceptional $\varepsilon_D, \varepsilon_D^4$ case occurs only when $D = 1785$ or $D = 28560$.*

Now our case is $D = 3$. Since $\varepsilon_3 = 2 + \sqrt{3}, \varepsilon_3^2 = 7 + 4\sqrt{3}$, we know $b_k = b_1 = 1$ and $b_{k+1} = b_2 = 2$. Thus we have verified $Y = 2m + 1 = a_1 + b_1 = 2 + 1$,i.e., $m = 1$. Hence $P_3(m) = y^2 = P_5(n)$ if and only if the trivial case $m = n = 1$.

2.3 Integral Points on Modular Elliptic Curve

$$P_3(m) = \frac{m(m+1)}{2} = y^2 = P_5(n) = \frac{n(3n-1)}{2}.$$

implies

$$\begin{cases} X^2 - 6Z^2 &= 1, \\ Y^2 - 2Z^2 &= 1, \end{cases}$$

where $X = 6n-1$, $Y = 2m+1$ and $Z = 2y$. Since $(XY)^2 = (2Z^2+1)(6Z^2+1)$, we obtain

$$(12Z)^2 \times (XY)^2 = (12Z^2)(12Z^2 + 2)(12Z^2 + 6).$$

Putting $y = 12XYZ$ and $x = 12Z^2 + 3$, we have the modular elliptic curve

$$E : y^2 = x^3 - x^2 - 9x + 9.$$

We note that each pentagonal square triangular number corresponds to an integer point on this elliptic curve E. This curve's Cremona label is 192A2(R) and the Mordell Weil group $E(\mathbb{Q})$ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$. The generators of $E(\mathbb{Q})$ are given by

$$\mathbb{Z} = \langle P_1 = (0, 3)\rangle, \mathbb{Z}/2\mathbb{Z} = \langle P_2 = (3, 0)\rangle, \mathbb{Z}/2\mathbb{Z} = \langle P_3 = (1, 0)\rangle.$$

Then all the integral points on E are given by

$$E(\mathbb{Z}) = \{(-5, \pm 8), (-3, 0), (-1, \pm 4), (0, \pm 3), (1.0), (3, 0), (9, \pm 24), (51, \pm 360)\}.$$

$x = 12Z^2 + 3$ yields $X = 5, Y = 3, Z = 2$, i.e., $m = y = n = 1$. Hence we have verified the following theorem by three different approaches.

Theorem 2.3 *There exists no pentagonal square triangular number $P_3(m) = y^2 = P_5(n)$ except for the trivial case $P_3(1) = 1^2 = P_5(1)$.*

Remark 2.4 *The first author gave a detailed explanation of the determination of the integer points $E(\mathbb{Z})$ of the above modular elliptic curve E in his paper* [12]*(section 3).*

3 Heptagonal Square Triangular Number

3.1 Square Terms in Binary Recurrence Sequences

Assume

$$P_3(m) = \frac{m(m+1)}{2} = y^2 = P_7(n) = \frac{n(5n-3)}{2}.$$

Then we have

$$\begin{cases} (2m+1)^2 - 8y^2 & = & 1, \\ (10n-3)^2 - 40y^2 & = & 9. \end{cases} \iff \begin{cases} (2m+1)^2 - 8y^2 & = & 1, \\ (10n-3)^2 - 5(2m+1)^2 & = & 4. \end{cases}$$

Let us denote kth Fibonacci and Lucas number by F_k and L_k, respectively. Then $(10n-3)^2 - 5(2m+1)^2 = 4$ implies $10n - 3 = L_{2k}$ and $2m+1 = F_{2k}$ for some positive integer k. Then the following identity is well known

$$(8y^2) = F_{2k}^2 - 1 = F_{2k+2}F_{2k-2} = F_{k+1}L_{k+1}F_{k-1}L_{k-1}.$$

Then $(F_{k+1}, L_{k+1}) = (F_{k+1}, F_{k-1}) = (L_{k-1}, F_{k-1}) = 1$ from the definition.

$$\frac{L_{k+1} + F_{k+1}\sqrt{5}}{2} = \left(\frac{L_{k-1} + F_{k-1}\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{5}}{2}\right)$$

implies

$$2L_{k+1} = 3L_{k-1} + 5F_{k-1}, 2F_{k+1} = L_{k-1} + 3F_{k-1}.$$

Hence

$$(L_{k+1}, F_{k-1}) \mid (2L_{k+1}, F_{k-1}) = (3L_{k-1}, F_{k-1}) = (3, F_{k-1}),$$

and

$$(L_{k-1}, F_{k+1}) \mid (L_{k-1}, 2F_{k+1}) = (L_{k-1}, 3F_{k-1}) = (3, L_{k-1}).$$

$(F_{k-1}, L_{k-1}) = 1$ implies at least one of (L_{k+1}, F_{k-1}) and (L_{k-1}, F_{k+1}) is 1. Since $F_{k-1}L_{k-1}F_{k+1}L_{k+1} = 8y^2$, we have $F_{k-1} = \square$ or $2\square$ or $F_{k+1} = \square$ or $2\square$. From J. H. E. Cohn's classical results on square and two times square in Fibonacci numbers, $F_i = \square, 2\square$ if and only if $i = 0, 1, 2, 3, 6, 12$. Hence $k \in \{2, 3, 4, 5, 7, 11, 13\}$. Trying one by one, we see $k = 2$ is the only the case $F_{2k}^2 - 1 = 8\square$, which yields the case $m = n = 1$.

3.2 Integral Points on Modular Elliptic Curve

Since $(2m+1)^2 = 8y^2 + 1$ and $(10n-3)^2 = 40y^2 + 9$, we obtain $5 \times (2m+1)^2(10n-3)^2 = (40y^2+5)(40y^2+9)$. Then we have

$$80y^2 \times 2^2 \times 5(2m+1)^2(10n-3)^2 = 80y^2(80y^2+10)(80y^2+18).$$

Putting $Y = 40(2m+1)(10n-3)y$ and $X = 80y^2 + 9$, we have the modular elliptic curve

$$E : Y^2 = X^3 + X^2 - 81X - 81.$$

We note that any heptagonal square triangular number corresponds to an integer point on this elliptic curve E. This curve's Cremona label is 960j2 and the Mordell Weil group $E(\mathbb{Q})$ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$. Then the generators of $E(\mathbb{Q})$ are given by

$$\mathbb{Z} = \langle P_1 = (6, -15)\rangle, \mathbb{Z}/2\mathbb{Z} = \langle P_2 = (-1, 0)\rangle, \mathbb{Z}/2\mathbb{Z} = \langle P_3 = (9, 0)\rangle.$$

Then all the integral points on E are given by

$$E(\mathbb{Z}) = \{(-1, 0), (-9, 0), (-6, \pm 15), (-3, \pm 12, (15, \pm 48), (39, \pm 240), (89, \pm 840)\}.$$

$X = 80y^2 + 9$ yields $y = 1$ and $10n - 3 = 7, 2m + 1 = 3$, i.e., $m = y = n = 1$. Hence we have verified the following theorem by two different approaches.

Theorem 3.1 *There exists no heptagonal square triangular number $P_3(m) = y^2 = P_7(n)$ except for the trivial case $P_3(1) = 1^2 = P_7(1)$, that is,*

$$P_3 \cap P_4 \cap P_7 = \{1\}.$$

4 Several Polygonal Square Triangular Numbers

4.1 Octagonal Square Triangular Number

Assume

$$P_3(m) = \frac{m(m+1)}{2} = y^2 = P_8(n) = n(3n-2).$$

Hence

$$\begin{cases} (2m+1)^2 &= 8y^2 + 1, \\ (3n-1)^2 &= 3y^2 + 1. \end{cases}$$

Thus

$$24^2 y^2 \times (2m+1)^2 (3n-1)^2 = (24y^2)(24y^2 + 3)(24y^2 + 8).$$

Putting $X = 24y^2 + 4$ and $Y = 24(2m+1)(3n-1)$, we have the modular elliptic curve

$$E : Y^2 = X^3 - X^2 - 16X + 16.$$

Then the Cremona label of E is 240c2 and the Mordel Weil group $E(\mathbb{Q})$ and the generators are $E(\mathbb{Q}) \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$, and
$\mathbb{Z} = \langle P_1 = (-2, 6)\rangle, \mathbb{Z}/2\mathbb{Z} = \langle P_2 = (1, 0)\rangle, \mathbb{Z}/2\mathbb{Z} = \langle P_3 = (4, 0)\rangle$.
Hence all the integral points on E are
$E(\mathbb{Z}) = \{(-4, 0), (-2, \pm 6), (0, \pm 4), (1, 0), (4, 0), (5, \pm 6), (6, \pm 10), (16, \pm 60), (28, \pm 144), (246, \pm 3850)\}$. Since $X = 24y^2 + 4$, we know $X = 28$, i.e., $y = 1$ and hence $m = n = 1$.

Theorem 4.1

$$P_3 \cap P_4 \cap P_8 = \{1\}.$$

4.2 Other Polygonal Square Triangular numbers

Put $P_3(m) = \dfrac{m(m+1)}{2} = y^2 = P_9(n) = \dfrac{n(7n-5)}{2}$. Then

$$\begin{cases} (2m+1)^2 &= 8y^2+1, \\ (14n-5)^2 &= 56y^2+25. \end{cases}$$

Thus

$$56^2y^2 \times (2m+1)^2(14n-5)^2 = (112y^2)(112y^2+14)(114y^2+50).$$

Putting $X = 112y^2 + 21$ and $Y = 56(2m+1)(14n-5)$, we obtain the elliptic curve

$$E : Y^2 = X^3 + X^2 - 665X + 4263.$$

Then the Cremona label of E is 6720ck2 and the Mordell Weil group $E(\mathbb{Q})$ and the generators of $E(\mathbb{Q})$ are
$E(\mathbb{Q}) \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$ and
$\mathbb{Z} = \langle P_1 = (-14, 105)\rangle, \mathbb{Z}/2\mathbb{Z} = \langle P_2 = (7, 0)\rangle, \mathbb{Z}/2\mathbb{Z} = \langle P_3 = (21, 0)\rangle$.
Therefore the integral points $E(\mathbb{Z})$ are given by
$E(\mathbb{Z}) = \{(-29, 0), (-14, \pm 105), (1, \pm 60), (3, \pm 48), (7, 0), (21, 0), (31, \pm 120), (91, \pm 840), (133, \pm 1512), (2911, \pm 157080)\}$. Since $X = 112y^2 + 21$, we know $X = 133$, i.e., $y = 1$ and hence $m = n = 1$.

Theorem 4.2

$$P_3 \cap P_4 \cap P_9 = \{1\}.$$

In section 5, we shall show the following two theorems in more general settings.

Theorem 4.3

$$P_3 \cap P_4 \cap P_{10} = \{1\}.$$

Theorem 4.4

$$P_3 \cap P_4 \cap P_{11} = \{1\}.$$

Put $P_3(m) = \dfrac{m(m+1)}{2} = y^2 = P_{12}(n) = n(5n-4)$ Then

$$\begin{cases} (2m+1)^2 &= 8y^2+1, \\ (5n-2)^2 &= 5y^2+4. \end{cases}$$

Thus

$$5^2 \times 8^2y^2 \times (2m+1)^2(5n-2)^2 = (40y^2)(40y^2+5)(40y^2+32).$$

Putting $X = 40y^2 + 12$ and $Y = 40(2m+1)(5n-2)$, we obtain the elliptic curve

$$E_1 : Y^2 = X^3 + X^2 - 296X + 1680.$$

Then the minimal Weierstrass equation of E_1 is

$$E_2 : y^2 + xy + y = x^3 - 19x + 26.$$

Cremona label of E_2 is 30a2 and the Mordell Weil group of E_2 has rank 0, i.e., $E_2(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \cong E_1(\mathbb{Q})$. Then the generators of the torsion group of E_1 are

$$\mathbb{Z}/2\mathbb{Z} = \langle P_1 = (12,0)\rangle, \mathbb{Z}/6\mathbb{Z} = \langle P_2 = (-8,60)\rangle.$$

Thus there are 11 rational(and so integral) points $E(\mathbb{Z}) = \{P_2 = (-8,60),$
$2P_2 = (16,-36), 3P_2 = (7,0), 4P_2 = (16,36), 5P_2 = (-8,-60), P_1 = (12,0),$
$P_1 + P_2 = (4,-24), P_1 + 2P_2 = (52,360), P_1 + 3P_2 = (-20,0),$
$P_1 + 4P_2 = (52,-360), P_1 + 5P_2 = (4,24)\}$. Since $X = 40y^2 + 12$, we obtain $X = 52$, i.e., $y = 1$ and hence $m = n = 1$.

Theorem 4.5

$$P_3 \cap P_4 \cap P_{12} = \{1\}.$$

5 Miscellaneous

5.1 Trivial Cases

It was Euler who has shown the infinity of the square triangular numbers. Here we refer to the several cases when the set of the polygonal square number is finite. Let $P_{10}(m)$ be the decagonal square number, that is,

$$P_{10}(m) = 4m^2 - 3m = y^2.$$

Then $16y^2 = (8m-3)^2 - 9$ implies

$$(8m - 3 + 4y)(8m - 3 - 4y) = 9.$$

Hence $8m - 3 + 4y = 9$ and $8m - 3 - 4y = 1$. Therefore $y = 1$ and $m = 1$, i.e., the only decagonal square number is the trivial case $P_{10}(1) = 1^2 = 1$.
Put $k = 2a^2+2$, where a is a natural number. Then $P_k(m) = m(a^2m-(a^2-1))$ and one can easily verify the finiteness of the set of $k-$gonal square numbers as follows. Since $P_k(m) = m(a^2m - (a^2-1))$, $P_k(m) = y^2$ implies $4a^2y^2 = 4a^2m(a^2m - (a^2-1)) = (2a^2m - (a^2-1))^2 - (a^2-1)^2$. Then

$$(2a^2m - a^2 + 1 + 2ay)(2a^2m - a^2 + 1 - 2ay) = (a^2-1)^2.$$

Since there are only finitely many decompositions of $AB = (a^2-1)^2$ into the natural numbers A, B, we know the finiteness of the set $\{P_k(m) = y^2 \mid m, y \in \mathbb{N}\}$.

Proposition 5.1 *Let a be a natural number and $k = 2a^2 + 2$. Then there exist only finitely many $k-$gonal square number.*

Remark 5.2 *We note that the decagonal square number is nothing but the case $a = 2$. We also obtain $\{P_{20}(m) = y^2 \mid m, y \in \mathbb{N}\} = \{1\}$ for the case $a = 3$, and $\{P_{34}(m) = y^2 \mid m, y \in \mathbb{N}\} = \{1, 14^2\}$ for the case $a = 4$.*

Here we note some of the simultaneous diophantine equations can be solved by elementary factorizations as follows. Consider the following simultaneous diophantine equations

$$\begin{cases} x^2 - Ry^2 &= C, \\ z^2 - Sy^2 &= D, \end{cases}$$

where R, S are not square positive integers. Assume $RS = \square = M^2$. Then

$$S^2x^2 - RSz^2 = (Sx + Mz)(Sx - Mz) = CS^2 - RSD.$$

Thus the solutions of the above diophantine equations are determined by the factorizations of $S(CS - RD)$, and hence the number of solutions is finite. For example, assume

$$P_3(m) = y^2 = P_{11}(n) = \frac{n(9n-7)}{2}.$$

Then

$$\begin{cases} (2m+1)^2 - 8y^2 &= 1, \\ (18n-7)^2 - 72y^2 &= 49. \end{cases}$$

Hence

$$((18n-7) + 3(2m+1))((18n-7) - 3(2m+1)) = 40.$$

From the decomposition $40 = 2 \times 20$ implies the only one positive integer solution $m = n = 1$.

Assume $k = a^2 + 2$. Then $8 \times a^2 P_k(n) = (2a^2n - (a^2-2))^2 - (a^2-2)^2$. Therefore it is $R = 8$ and $S = 8a^2$ case of the above diophantine equation. Therefore we can determine the solutions m, n by simple factorizations of $(a^2 - 2)^2 - a^2 = (a-1)(a+1)(a-2)(a+2)$.

We can generalize the above results as follows.

Proposition 5.3 *In the case $k = a^2 + 2$, one can determine the solutions of $P_3(m) = y^2 = P_k(n)$ by the factorizations of $(a-1)(a+1)(a-2)(a+2)$.*

5.2 Several other cases

Assume

$$P_5(m) = \frac{m(3m-1)}{2} = y^2 = P_8(n) = n(3n-2).$$

Then we have

$$\begin{cases} (6m-1)^2 - 24y^2 &= 1, \\ (3n-1)^2 - 3y^2 &= 1. \end{cases}$$

$$72^2y^2 \times (6m-1)^2(3n-1)^2 = (72y^2)(72y^2+3)(72y^2+24).$$

Put $Y = 72y(6m-1)(3n-1)$ and $X = 72y^2 + 9$. Then we have the modular elliptic curve

$$E : Y^2 = X^3 - 171x + 810.$$

This curve's Cremona label is 1008h2 and the Mordell Weil group $E(\mathbb{Q})$ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$. Then the generators of $E(\mathbb{Q})$ are given by

$$\mathbb{Z} = \langle P_1 = (-3, 36)\rangle, \mathbb{Z}/2\mathbb{Z} = \langle P_2 = (6, 0)\rangle, \mathbb{Z}/2\mathbb{Z} = \langle P_3 = (9, 0)\rangle.$$

Thus all the integral points on E are given by
$E(\mathbb{Z}) = \{(-15, 0), (-3, \pm 36), (3, \pm 18), (6, 0), (9, 0), (10, \pm 10), (13.\pm 28), (27, \pm 126), (81, \pm 720)\}$. Then $X = 72y^2 + 9$ yields $X = 81, y = 1$, i.e., $m = y = n = 1$. Hence we have verified the following theorem.

Theorem 5.4

$$P_4 \cap P_5 \cap P_8 = \{1\}.$$

Assume

$$P_3(m) = \frac{m(3m-1)}{2} = P_5(n) = \frac{n(3n-1)}{2} = P_8(k) = n(3k-2).$$

Put $y = 6n - 1$, we have

$$\begin{cases} 3(2m+1)^2 &= y^2 + 2, \\ 8(3k-1)^2 &= y^2 + 7. \end{cases}$$

$$6^3 \times 24 \times y^2 \times (2m+1)^2(3k-1)^2 = (6y^2)(6y^2+12)(6y^2+42).$$

Put $Y = 72y(2m+1)(3k-1)$ and $X = 6y^2 + 18$, we have the modular elliptic curve

$$E : Y^2 = X^3 - 468x + 2512.$$

This curve's Cremona label is 20160lg2 and the Mordell Weil group $E(\mathbb{Q})$ is isomorphic to $\mathbb{Z}^2 \times (\mathbb{Z}/2\mathbb{Z})^2$. Then the generators of $E(\mathbb{Q})$ are given by
$\mathbb{Z} = \langle P_1 = (-18, 72)\rangle, \mathbb{Z} = \langle P_2 = (-14, 80)\rangle, \mathbb{Z}/2\mathbb{Z} = \langle P_3 = (6, 0)\rangle$, and $\mathbb{Z}/2\mathbb{Z} = \langle P_4 = (18, 0)\rangle$. Hence all the integral points on E are given by
$E(\mathbb{Z}) = \{(-24, 0), (-18, \pm 72), (-14, \pm 80), (-6, \pm 72), (-3, \pm 63), (4, \pm 28), (6, 0), (18, 0), (21.\pm 45), (36, \pm 180), (46, \pm 280), (102, \pm 1008), (168, \pm 2160), (186, \pm 2560), (381, \pm 7425), (1476, \pm 56700), (2034, \pm 91728), (67246, \pm 17438120)\}$.
$X = 6y^2 + 18$ yields $X = 24$ i,e., $y = 1$, or $X = 168$ i.e., $y = 5$. From the condition $y = 6n - 1$, we know $y \neq 1$. Hence $y = 6n - 1 = 5$, i.e., $m = y = n = 1$. Hence we have verified the following theorem.

Theorem 5.5

$$P_3 \cap P_5 \cap P_8 = \{1\}.$$

References

[1] W. S. Anglin, The Queen of Mathematics, Kluwer Academic Publishers, Boston 1995.

[2] W. S. Anglin, Simultaneous Pell equations, *Math. Comput.*, **65**, No. 213 (1996), 355-359.

[3] A. Baker and H. Davenport, The equations $3x^2-2=y^2$ and $8x^2-7=z^2$, *Quat. J. Math. Oxford* 2(2), **20** (1969), 127-137.

[4] M. A. Bennett, On the number of solutions of simultaneous Pell equations. J. Reine Angew. Math., **498** (1998), 173-199.

[5] M. A. Bennett, M. Cipu, M. Mignotte and R. Okazaki, On the number of solutions of simultaneous Pell equations II, *Acta Arith.*, **122**, No. 4 (2006), 407-417.

[6] J. H. E. Cohn, On square Fibonacci numbers, *J. Lond. Math. Soc.*, **39** (1964), 537-540.

[7] J. H. E. Cohn, The Diophantine equation $x^4 - Dy^2 = 1$, II, *Acta Arith.*, **59** (1997), 401-403.

[8] J. E. Cremona, Algorithms for Modular Elliptic Curves (2nd ed.), Cambridge Univ. Press, Cambridge 1997.

[9] E. Deza, Figurate Numbers, World Scientific Publishing, 2012.

[10] L. E. Dickson, History of the theory of numbers vol. II, Dover Publications, 1970[1920].

[11] J. Gebel, A Pethő and H. Zimmer, Computing integral points on elliptic curves, *Acta Arith.*, **68** (1994), 171-192.

[12] S. Katayama, Several methods for solving simultaneous Fermat-Pell equations, *J. Math. Tokushima Univ.*, **33** (1999), 1-14.

[13] T. Koshy, Pell and Pell-Lucas Numbers with Applications, Springer, New York 2014.

[14] W. Ljunggren, Einige Eigenschaften der Einheiten reeller quadraticher und reinbiquadraticher Zahnköper, *Oslo Vid. Akad. Strifter*, **1**, No. 12 (1936).

[15] The LMFDB Collaboration, The L-functions and Modular Forms Database, http://www.lmfdb.org, 2013, [On-line; accessed 17 September 2020].

[16] L. J. Mordell, Diophantine Equations, Academic Press, London and New York 1969.

[17] The On-Line Encyclopedia of Integer Sequences, http://www.oeis.org, [On-line; accessed 17 September 2020].

[18] P. Ribenboim and W. L. McDaniel, The squares in Lucas sequences, J. Number Theory, **58** (1996), 104-123.

[19] J. H. Rickert, Simultaneous rational approximations and related diophantine equations, *J. Math. Cambridge Philos. Soc.*, **113** (1993), 461-472.

[20] Sage Mathematics Software (Version 9.1), 2020, http://www.sagemath.org.

[21] J. H. Silverman, Friendly Introduction to Number Theory, Addison Wesley, 2012.

[22] T. Takagi, Lectures on Elementary Number Theory(2nd ed), Kyoritsu P Shuppan, 1971.

[23] M. Waldschmidts, A lower bound for linear forms in logarithms, *Acta Arith.*, **37** (1980), 257-283.

[24] Wolfram Mathworld, Pentagonal Triangular Number, https://mathworld.wolfram.com/PentagonalTriangularNumber.html, [On-line; accessed 17 September 2020].