J. Math. Tokushima Univ.
Vol. 54 (2020), 41 − 55

Axiomatic Method of Measure and Integration (VI). Definition of the RS-integral and its Fundamental Properties

(Yoshifumi Ito "RS-integral and LS-integral", Chapter 4)

By

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(Received September 30, 2020)

Abstract

In this paper, we define the RS-integral of the RS-measurable functions on $\boldsymbol{R}^d$, $(d \geq 1)$.

Then we study the fundamental properties of the RS-integral. These facts are the new results.

2000 Mathematics Subject Classification. Primary 28A25.

Introduction

This paper is the part VI of the series of the papers on the axiomatic method of measure and integration on the Euclidean space. As for the details, we refer to Ito [6], [14]. Further we refer to Ito [1] $\sim$ [5], [7] $\sim$ [13] and [15] $\sim$ [22].

In this paper, we study the definition of the d-dimensional RS-integral on the d-dimensional Euclidean space $\boldsymbol{R}^d$ and their fundamental properties. Here we assume $d \geq 1$. In the sequel, we happen to omit the adjective "d-dimensional".

We assume that the d-dimensional Euclidean space $\boldsymbol{R}^d$ is the d-dimensional RS-measure space $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$. Then we define the class of RS-measurable functions adapting to this RS-measure and we define the RS-integral for these RS-measurable functions.

Then, as the condition that a function $f(x)$ is RS-measurable, we have the possibility that we use the condition such that the level set $\{x;\ f(x) < \alpha\}$ is a RS-measurable set similarly to the Lebesgue theory. But, because, even though the RS-measure space $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ is a conditionally completely additive measure space, it is a finite additive measure space without any condition, we cannot construct the effective theory of integration by using the condition such that the level set $\{x;\ f(x) < \alpha\}$ is a RS-measurable set for an arbitrary real number α.

Therefore, in this paper, we define that $f(x)$ is RS-measurable if it is the limit of the direct family of the simple functions in the sense of uniform convergence in the wider sense on the set which does not include the singular points of $f(x)$.

Because the RS-integral is defined for the RS-measurable function, we must prove that the RS-measurability of the functions is preserved for the operations such as the four fundamental rules of calculation and the supremum, the infimum and the limit of functions in order to study the relations of the RS-integral and the operations of functions.

We give these results as the theorems of the properties of the measurable functions.

We define the RS-integrals for these RS-measurable functions and study their fundamental properties.

Then the concept of the uniform convergence in the wider sense on the set excluding the singular points of the integrands conforms well with the class of the RS-measurable functions and the class of the RS-integrable functions. Namely the function, which is the limit of the sequence of functions in this class in the sense of uniform convergence in the wider sense on the set excluding the singular points, belongs to the same class.

In the theory of RS-integral, the reason why we assume that the integration domain E is a RS-measurable set is as follows. If we assume that the considered subset E of $\boldsymbol{R}^d$ is not a RS-measurable set, even a constant function defined on E is not a RS-measurable function after all, it is meaningless in itself that we consider the RS-measurable functions on such a set E.

Thus the definition of the RS-integral on a RS-non-measurable set E is meaningless.

Here I express my heartfelt gratitude to my wife Mutuko for her help of typesetting of the TEX-file of this manuscript.

1 d-dimensional RS-measurable functions

In this section, we define the concept of d-dimensional RS-measurable functions and study their fundamental properties. Here we assume $d \geq 1$. In the sequel, we happen to omit the adjective"d-dimensional".

We assume that the Euclidean space $\boldsymbol{R}^d$ is the d-dimensional RS-measure space $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$.

Then we assume that a subset E of the Euclidean space $\boldsymbol{R}^d$ is a **RS-measurable set**. In the sequel, for the simplicity, we say that E is measurable.

Now we decompose a set E to the direct sum of a countable measurable subsets $E_1,\ E_2,\ E_3,\ \cdots$. Namely, every pair of $E_1,\ E_2,\ E_3,\ \cdots$ are mutually disjoint and their sum is equal to E. We denote this as follows:

$$(\Delta) : E = E_1 + E_2 + E_3 + \cdots .$$

In the sequel, we consider that every countable direct sum decomposition by using the measurable subsets of E. We say simply that this is a **division** of E. If we have these two divisions Δ and Δ' of E, we say that Δ' is a **fine division** of Δ if each small subset of Δ' is included in a certain small subset of Δ.

By dividing a certain small subset of the division Δ to two subsets, we have a fine division of Δ. A general fine division of a division Δ is obtained by a countable iteration of the fine division by virtue of such dividing into two parts.

If Δ' is a fine division of Δ for two divisions Δ and Δ' of E, we define the order and denote it as

$$\Delta \leq \Delta'.$$

This is a semiorder.

Now we denote the all divisions of E as $\boldsymbol{\Delta} = \boldsymbol{\Delta}(E)$. This is a **direct set** by virtue of the semiorder defined by the fine division.

Namely this means that we have the following conditions (1) $\sim$ (4).

Here, assume that $\Delta,\ \Delta',\ \Delta'' \in \boldsymbol{\Delta}$ hold.

(1) For every $\Delta \in \boldsymbol{\Delta}$, we have $\Delta \leq \Delta$.

(2) If we have $\Delta \leq \Delta'$ and $\Delta' \leq \Delta$, we have $\Delta = \Delta'$.

(3) If we have $\Delta \leq \Delta'$ and $\Delta' \leq \Delta''$, we have $\Delta \leq \Delta''$.

(4) For any two elements $\Delta,\ \Delta' \in \boldsymbol{\Delta}$, there exists a certain $\Delta'' \in \boldsymbol{\Delta}$ such that we have $\Delta \leq \Delta''$ and $\Delta' \leq \Delta''$.

Here we give the definition of a simple function.

Definition 1.1 We say that a function $f(x)$ defined on a measurable set E is a **simple function** if $f(x)$ is expressed as

$$f(x) = \sum_{j=1}^{\infty} \alpha_j \chi_{E_j}(x) \tag{1.2}$$

for a division Δ of E:

$$(\Delta): \ E = E_1 + E_2 + E_3 + \cdots . \tag{1.3}$$

Here α_j is a real number or $\pm\infty$ and each pair of them has not to be mutually different. $\chi_{E_j}(x)$ denotes the defining function of the set E_j. Then we denote the simple function $f(x)$ as $f_\Delta(x)$.

The defining function $\chi_E(x)$ of the set E in Definition 1.1 is defined as follows:

$$\chi_E(x) = \begin{cases} 1, & (x \in E), \\ 0, & (x \notin E). \end{cases}$$

Because a simple function $f(x)$ is a function, its range is determined. Namely, for such a simple function $f(x)$, its range is at most countable set and the possible values are at most countable real numbers or $\pm\infty$.

But the way of the expression of the form (1.2) for a simple function $f(x)$ has an infinitely many varieties. It is the reason why the form of division (1.3) of E has an infinitely many varieties. Especially we say that a simple function for a finite division of a measurable set E is a **step function**.

Now we assume that $\overline{\boldsymbol{R}} = [-\infty, \infty]$ is the **extended real number space**.

Here we assume that a function $f(x)$ on E considered here takes its value in $\overline{\boldsymbol{R}}$. We say that this function is an **extended real-valued function**. Then we denote $E(\infty) = \{ x \in E;\ |f(x)| = \infty \}$. We say that a point in the set $E(\infty)$ is a **singular point** of $f(x)$.

Then we define a measurable function as follows.

Definition 1.2 We say that an extended real-valued function $f(x)$ defined on a measurable set E of $\boldsymbol{R}^d$ is a **RS-measurable function** if it satisfies the following conditions (i) and (ii):

(i) We have $E(\infty) \in \mathcal{M}_0$ and $\mu(E(\infty)) = 0$.

(ii) There exist a direct family $\{ f_\Delta(x);\ \Delta \in \boldsymbol{\Delta} = \boldsymbol{\Delta}(E) \}$ of simple functions such that we have the limit

$$\lim_{\Delta} f_\Delta(x) = f(x) \tag{1.4}$$

in the sense of the uniform convergence in the wider sense on $E \backslash E(\infty)$.

The condition (ii) of Definition 1.2 means that the following condition (iii) holds:

(iii) For an arbitrary measurable bounded closed set K included in $E \backslash E(\infty)$ and for an arbitrary positive number $\varepsilon > 0$, there exists a certain $\Delta_0 \in \mathbf{\Delta}$ such that, for an arbitrary $\Delta \in \mathbf{\Delta}$ so that $\Delta_0 \leq \Delta$ holds, we have the inequality

$$|f_\Delta(x) - f(x)| < \varepsilon, \ (x \in K). \tag{1.5}$$

We happen to say that the limit in the sense of the formula (1.4) is the **Moore-Smith limit**. Intuitively, this means that this is the limit when we iterate the division of E infinitely many times and continue the division of E so that the division becomes finer, finer and infinitely finer.

For simplicity, we say that a RS-measurable function $f(x)$ is a **measurable function** or **measurable**.

For one measurable function $f(x)$, we have an infinitely many varieties of the choices of the direct families $\{f_\Delta(x);\ \Delta \in \mathbf{\Delta}\}$ of simple functions which satisfy the condition (1.4).

When we fix one direct set $\mathbf{\Delta} = \mathbf{\Delta}(E)$, this variety depends on the way of choice of the range of a simple function $f_\Delta(x)$.

We say that a direct subset $\mathbf{\Delta}'$ of $\mathbf{\Delta} = \mathbf{\Delta}(E)$ is cofinal with $\mathbf{\Delta}$ if, for an arbitrary $\Delta \in \mathbf{\Delta}$, there exist $\Delta' \in \mathbf{\Delta}'$ such that $\Delta \leq \Delta'$ holds.

For a certain division $\Delta_0 \in \mathbf{\Delta}$, we put $\mathbf{\Delta}' = \{\Delta' \in \mathbf{\Delta};\ \Delta' \geq \Delta_0\}$. Then $\mathbf{\Delta}'$ is a direct subset which is cofinal with $\mathbf{\Delta}$.

We assume that a direct subset $\mathbf{\Delta}'$ of $\mathbf{\Delta}$ is cofinal with $\mathbf{\Delta}$.

Then the following (I) and (II) are equivalent:

(I) A direct family $\{f_\Delta(x);\ \Delta \in \mathbf{\Delta}\}$ of simple functions on E converges to a function $f(x)$ on E at $E \backslash E(\infty)$:

(II) A direct subfamily $\{f_{\Delta'}(x);\ \Delta' \in \mathbf{\Delta}'\}$ of simple functions on E converges to a function $f(x)$ on E at $E \backslash E(\infty)$:

in the sense of one of the following (1) $\sim$ (3):

(1) the point-wise convergence,

(2) the uniform convergence,

(3) the uniform convergence in the wider sense.

Example 1.1 Assume that E is a measurable set in $\boldsymbol{R}^d$. Then a simple function $f(x)$ and a continuous function $f(x)$ defined on E are measurable.

Theorem 1.1 *Assume that E is a measurable set in $\boldsymbol{R}^d$. Further assume that two functions f and g are two measurable functions defined on E.*

Then the following functions (1) $\sim$ (10) *are the measurable functions defined on* E:

(1) $f+g$. (2) $f-g$. (3) fg. (4) f/g. *Here we assume that, for each measurable bounded closed set* F *in* E, *there exists a certain constant* $\delta > 0$ *such that* $|g| \geq \delta$ *holds on* F.

(5) αf. *Here we assume that* α *is a real constant.*

(6) $|f|$. (7) $\sup(f,\ g)$. (8) $\inf(f,\ g)$. (9) $f^+ = \sup(f,\ 0)$. (10) $f^- = -\inf(f,\ 0)$.

We define the functions $\sup(f,\ g)$ and $\inf(f,\ g)$ appeared in Theorem 1.1 in the following:

$$\sup(f,\ g)(x) = \sup(f(x),\ g(x)),\ (x \in E),$$
$$\inf(f,\ g)(x) = \inf(f(x),\ g(x)),\ (x \in E).$$

Further we have the formulas

$$f = f^+ - f^-,\ |f| = f^+ + f^-.$$

Theorem 1.2 *Assume that* E *is a measurable set in* $\boldsymbol{R}^d$. *Assume that a sequence* $\{f_n\}$ *of functions is a sequence of measurable functions on* E *and it converges to a function* f *at* $E \backslash E(\infty)$ *in the sense of uniform convergence in the wider sense. Then the function* f *is a measurable function on* E.

Thus the class of all measurable functions on E is closed with respect to the uniform convergence in the wider sense excluding the singular points.

Especially, for the Jordan measure space $(\boldsymbol{R}^d,\ \mathcal{B},\ \mu)$, we have the Example 1.2 and the Remark 1.1 in the following.

Because the Jordan measure space is a special example of the RS-measure space, in general, the concept of convergence in the sense of point wise convergence is not appropriate in order to study a RS-measurable functions.

Therefore, when we study the general RS-measurable functions, we need to use the concept of convergence in the sense of uniform convergence or uniform convergence in the wider sense.

Example 1.2 We assume that the function $f(x)$ defined on the interval $[0,\ 1]^d$ is equal to 1 at the rational point and it is equal to 0 at the other points. Then $f(x)$ is not a measurable function on $[0,\ 1]^d$.

Remark 1.1 For the function $f(x)$ in Example 1.2, we have the equality

$$f(x) = \lim_{m_1,\ \cdots,\ m_d \to \infty}\ \lim_{n_1,\ \cdots,\ n_d \to \infty} \prod_{j=1}^{d} \{\cos m_j! \pi x_j\}^{2n_j},\ (x \in [0,\ 1]^d).$$

Then, since the function

$$\prod_{j=1}^{d} \{ \cos m_j! \pi x_j \}^{2n_j}$$

is continuous, it is Jordan measurable. But the limit function of this sequence of functions in the sense of point wise convergence is equal to the function $f(x)$ in Example 1.2 and this function $f(x)$ is not Jordan measurable.

Therefore the concept of convergence in the sense of point-wise convergence is not approximate in order to study the Jordan measurable functions. Hence, when we study the Jordan measurable functions, we can understand the meaning of using the concept of convergence in the sense of uniform convergence or in the sense of uniform convergence in the wider sense.

2 Definition of the *d*-dimensional RS-integral

In this section, we assume that the Euclidean space $\boldsymbol{R}^d$ is the d-dimensional RS-measure space $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ for $d \geq 1$.

Then we define the concept of the d-dimensional RS-integral for a measurable function. Here we consider the RS-integral of a function of d-variables. We consider that the integration domain is a d-dimensional set which is not necessarily an interval.

We assume that a subset E of the Euclidean space $\boldsymbol{R}^d$ is a measurable set. Further we assume that a function $f(x)$ is a measurable function defined on E.

Then we define the RS-integral of $f(x)$ in the following two steps.

(1) Case where $f(x)$ is a simple function

Then, for a division of E

$$(\Delta) : E = E_1 + E_2 + E_3 + \cdots , \tag{2.1}$$

we assume that $f(x)$ is expressed as follows:

$$f(x) = \sum_{j=1}^{\infty} \alpha_j \chi_{E_j}(x). \tag{2.2}$$

Here we assume that α_j is a real number or $\pm\infty$, $(j \geq 1)$. Then we define the integral of $f(x)$ on E as the sum of the right hand side of the formula

$$\int_E f(x) d\mu = \sum_{j=1}^{\infty} \alpha_j \mu(E_j) \tag{2.3}$$

and we denote it by the symbol of the left hand side. Here we assume that the series in the right hand side converges absolutely.

The value of the right hand side of the formula (2.3) is determined uniquely and independently to the way of expression of the simple function $f(x)$ as the formula (2.2).

(2) Case where $f(x)$ is a general measurable function

Then there exists a direct family $\{f_\Delta(x);\ \Delta \in \boldsymbol{\Delta}\}$ of simple functions which converges to $f(x)$ on $E \backslash E(\infty)$ in the sense of uniform convergence in the wider sense. Here $\boldsymbol{\Delta}$ denotes the direct set of all divisions of E.

Then, if there exists a Moore-Smith limit

$$R = \lim_{\Delta} \int_E f_\Delta(x) d\mu,$$

we say that this limit R is the **RS-integral** of the function $f(x)$ on E and we denote it as

$$R = \int_E f(x) d\mu.$$

We assume that this value R does not depend on the choice of a direct family $\{f_\Delta(x);\ \Delta \in \boldsymbol{\Delta}\}$ of simple functions which converges to $f(x)$ on $E \backslash E(\infty)$ in the sense of uniform convergence in the wider sense.

The variety of the direct families $\{f_\Delta(x);\ \Delta \in \boldsymbol{\Delta}\}$ is the variety of the choices of the range of a simple function $f_\Delta(x)$.

Intuitively the Moore-Smith limit is considered as the limit when we continue the finite direct decomposition of E infinitely many times and continue the division of E so that the division becomes finer, finer and infinitely finer.

Now we study the relation to the classical definition of the Riemann type integral and we prove the Darboux Theorem.

Here we study the RS-integral in the case where, especially, E is a measurable bounded closed set of $\boldsymbol{R}^d$ and $f(x)$ is a bounded measurable function on E.

We assume that, for a finite division of E

$$(\Delta):\ E = F_1 + E_2 + \cdots + E_n,$$

a simple function $f_\Delta(x)$ is defined by the formula

$$f_\Delta(x) = \sum_{j=1}^{n} f(\xi_j)\chi_{E_j}(x),\ (\xi_j \in E_j,\ 1 \leq j \leq n).$$

Therefore, if there exists the RS-integral of $f(x)$ on E, the RS-integral R is equal to the Moore-Smith limit

$$R = \int_E f(x) d\mu = \lim_{\Delta} \int_E f_\Delta(x) d\mu = \lim_{\Delta} \sum_{j=1}^{n} f(\xi_j)\mu(E_j).$$

Namely the RS-integral R is equal to the Moore-Smith limit of the **Riemann sum**

$$R_\Delta = \sum_{j=1}^{n} f(\xi_j)\mu(E_j).$$

Especially when E is a closed interval

$$E = \prod_{j=1}^{n}[a_j,\ b_j],$$

we denote this RS-integral as

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_d}^{b_d} f(x)d\mu.$$

In general, even though we denote the integral on the higher dimensional space by using the symbol of the multi-integral, we denote the integral symbol by using one symbol without any permission because, in this paper, we consider the general framework such as the integration with respect to the measure μ.

Now we assume that E is a bounded measurable set and $f(x)$ is a bounded measurable function. Then we prove the Darboux Theorem.

We assume that we have one finite division of E

$$(\Delta) : E = E_1 + E_2 + \cdots + E_n.$$

Then we denote the supremum and the infimum of $f(x)$ on the subset E_i as M_i and m_i respectively and we denote the supremum and the infimum of $f(x)$ on the set E as M and m respectively. Now, if we put

$$h_\Delta(x) = \sum_{i=1}^{n} M_i\chi_{E_i}(x),$$
$$g_\Delta(x) = \sum_{i=1}^{n} m_i\chi_{E_i}(x),$$

we have the equalities

$$\lim_{\Delta} h_\Delta(x) = f(x),$$
$$\lim_{\Delta} g_\Delta(x) = f(x)$$

on E in the sense of uniform convergence in the wider sense because they are proved by the following way.

Because $f(x)$ is measurable, there exists a direct family $\{f_\Delta(x)\}$ of simple functions such that, for an arbitrary measurable bounded closed set K included in E and for an arbitrary positive number $\varepsilon > 0$, there exists a certain Δ_0 so that, for an arbitrary Δ such as $\Delta_0 \leq \Delta$, we have the inequality

$$|f_\Delta(x) - f(x)| < \varepsilon,\ (x \in K).$$

Therefore, if we have the equality

$$f_\Delta(x) = \sum_{j=1}^{n} \alpha_j \chi_{E_j}(x),$$

we have the inequality

$$\begin{aligned} &\alpha_j - \varepsilon < f(x) < \alpha_j + \varepsilon, \ (x \in K \cap E_j), \\ &(j = 1, \ 2. \ \cdots, \ n). \end{aligned}$$

Therefore, because ε is arbitrary, we have the inequalities

$$\begin{aligned} &|h_\Delta(x) - f(x)| < \varepsilon, \ (x \in K), \\ &|g_\Delta(x) - f(x)| < \varepsilon, \ (x \in K). \end{aligned}$$

Thereby we prove the assertion.

Now, we denote the integrals of $h_\Delta(x)$ and $g_\Delta(x)$ in the above on E as S_Δ and s_Δ respectively. Namely we put

$$\begin{aligned} S_\Delta &= \int_E h_\Delta(x) d\mu = \sum_{i=1}^{n} M_i \mu(E_i), \\ s_\Delta &= \int_E g_\Delta(x) d\mu = \sum_{i=1}^{n} m_i \mu(E_i). \end{aligned}$$

Then we have the inequalities

$$m\mu(E) \leq s_\Delta \leq S_\Delta \leq M\mu(E).$$

Here, because we have $\mu(E) < \infty$, the sets $\{s_\Delta; \ \Delta \in \mathbf{\Delta}\}$ and $\{S_\Delta; \ \Delta \in \mathbf{\Delta}\}$ are bounded. Therefore we have the equalities

$$S = \inf_\Delta S_\Delta, \ s = \sup_\Delta s_\Delta.$$

Therefore we have the inequality

$$s \leq S.$$

Then we have the following.

Theorem 2.1 (Darboux Theorem) *We use the notation in the above. Then we have the* Moore-Smith *limits*

$$S = \lim_\Delta S_\Delta, \ s = \lim_\Delta s_\Delta.$$

In the following, we study the integrability condition of a function $f(x)$. This is one of the existence theorems of the RS-integral.

Theorem 2.2 *We assume that E is a bounded measurable set in $\boldsymbol{R}^d$ and $f(x)$ is a bounded measurable function on E. Then $f(x)$ is integrable on E if and only if $s = S$ holds.*

We use the notation in the above. We say that $v_i = M_i - m_i$ is the oscillation of $f(x)$ on the subset E_i of the division Δ. Then, because we have the equality

$$V_\Delta = S_\Delta - s_\Delta = \sum_{i=1}^{n} v_i \mu(E_i),$$

the condition $s = S$ is equivalent to the condition

$$\lim_{\Delta} V_\Delta = \lim_{\Delta} \sum_{i=1}^{n} v_i \mu(E_i) = 0.$$

Thus we have the following.

Theorem 2.3 *A bounded measurable function $f(x)$ on a bounded measurable set E in $\boldsymbol{R}^d$ is integrable if and only if we have the equality*

$$\lim_{\Delta} V_\Delta = \sum_{i=1}^{n} v_i \mu(E_i) = 0.$$

Corollary 2.1 *We use the same notation as in Theorem 2.3. Then $f(x)$ is integrable on E if and only if, for an arbitrary positive number $\varepsilon > 0$, there exists a division Δ such that we have the inequality*

$$V_\Delta = \sum_{i=1}^{n} v_i \mu(E_i) < \varepsilon.$$

Proposition 2.1 *If a function $f(x)$ is integrable on a bounded measurable set E in $\boldsymbol{R}^d$, $f(x)$ is also integrable on an arbitrary measurable subset E' of E.*

As the existence theorem for a concrete function, we have the following Theorem 2.4. Then the condition that E is a bounded closed set is necessary for the proof of the Theorem 2.4 by using the condition of the uniform continuity of the continuous function.

Theorem 2.4 *A continuous function on a measurable bounded closed set E of $\boldsymbol{R}^d$ is integrable.*

Theorem 2.5 *We assume that E is a measurable bounded closed set of $\boldsymbol{R}^d$ and a function $f(x)$ is bounded measurable on E. If the measure of the set of discontinuous points of $f(x)$ is zero, $f(x)$ is integrable on E.*

In general, we assume that $(E,\ \mathcal{M}_0,\ \mu)$ is a d-dimensional RS-measure space and the RS-measure μ is defined by a variable-wise left continuous and continuous function of locally bounded variation $g(x)$.

Then we denote the RS-integral of a RS-integral function $f(x)$ on E as

$$\int_E f(x)d\mu = \int_E f(x)dg(x).$$

Further we use the same notation in the case of a variable-wise right continuous and continuous function of locally bounded variation $g(x)$.

Theorem 2.6 *We assume that a subset E of $\boldsymbol{R}^d$ is a* RS-*measurable set and $(E,\ \mathcal{M}_0,\ \mu)$ is a d-dimensional* RS-*measure space. Here we assume that the* RS-*measure μ is defined by a continuous function of locally bounded variation $g(x)$. We denote the positive variation and the negative variation of $g(x)$ as $g^+(x)$ and $g^-(x)$ respectively.*

Then $g^+(x)$ and $g^-(x)$ are two continuous functions and we have the equality

$$g(x) = g^+(x) - g^-(x).$$

Then there exists the RS-*integral of a continuous function $f(x)$ on E and we have the equality*

$$\int_E f(x)d(x) = \int_E f(x)dg^+(x) - \int_E f(x)dg^-(x).$$

In Theorem 2.6, we have the similar equalities in the case where $g(x)$ is a variable-wise left continuous function of locally bounded variation or in the case where $g(x)$ is a variable-wise right continuous function of loccally bounded variation.

3 Fundamental properties of the d-dimensional RS-integral

In this section, we assume that $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ is the RS-measure space on $\boldsymbol{R}^d$.

Theorem 3.1 *We assume that E is a bounded measurable set in $\boldsymbol{R}^d$ and a function $f(x)$ is a bounded and integrable on E. If $E = E_1 + E_2$ is a division of E, we have the following equality*

$$\int_E f(x)d\mu = \int_{E_1} f(x)d\mu + \int_{E_2} f(x)d\mu. \tag{3.1}$$

Theorem 3.2 *We assume that E is a bounded measurable set in $\boldsymbol{R}^d$ and two functions $f(x)$ and $g(x)$ are bounded and integrable on E. Then we have the following* (1) $\sim$ (3):

(1) *If α and β are two constants, $\alpha f(x) + \beta g(x)$ is also integrable on E and we have the following equality*

$$\int_E \{\alpha f(x) + \beta g(x)\}d\mu = \alpha \int_E f(x)d\mu + \beta \int_E g(x)d\mu. \tag{3.2}$$

(2) *$f(x)g(x)$ is also integrable on E.*

(3) *If there exists a constant k such that we have $|g(x)| \geq k > 0$ on E, $f(x)/g(x)$ is also integrable on E.*

Theorem 3.3 *We assume that $(E, \mathcal{M}_0,\ \mu)$ is a positive* RS-*measure space on E and two functions $f(x)$ and $g(x)$ are bounded and integrable on E. Then we have the following* (1) $\sim$ (3):

(1) *If we have $f(x) \geq 0$, we have the following inequality*

$$\int_E f(x)d\mu \geq 0. \tag{3.3}$$

Further, if $f(x)$ is continuous at an interior point $x_0 \in E$ and we have $f(x_0) > 0$, we have the following inequality

$$\int_E f(x)d\mu > 0. \tag{3.4}$$

(2) *If we have $f(x) \geq g(x)$, we have the following inequality*

$$\int_E f(x)d\mu \geq \int_E g(x)d\mu. \tag{3.5}$$

Further, if $f(x)$ and $g(x)$ are continuous at an interior point $x_0 \in E$ and we have $f(x_0) > g(x_0)$, we have the following inequality

$$\int_E f(x)d\mu > \int_E g(x)d\mu. \tag{3.6}$$

(3) $|f(x)|$ *is integrable on* E *and we have the following inequality*

$$|\int_E f(x)d\mu| \leq \int_E |f(x)|d\mu. \tag{3.7}$$

Theorem 3.4 (Theorem of the 1st mean-value of integral) *We assume that* E *is a bounded measurable set in* $\boldsymbol{R}^d$ *and* $(E,\ \mathcal{M}_0,\ \mu)$ *is a positive* RS-*measure space on* E. *We assume that a function* $f(x)$ *is bounded and integrable on* E *and we denote the supremum and the infmum of* $f(x)$ *on* E *as* M *and* m *respectively. Then there exists a real constant* α *such that we have the equality*

$$\int_E f(x)d\mu = \alpha\mu(E),\ m \leq \alpha \leq M. \tag{3.8}$$

Especially, if $f(x)$ *is continuous on a measurable bounded closed domain* E, *there exists a certain point* $x_0 \in E$ *such that we have* $\alpha = f(x_0)$.

Theorem 3.5 *We assume that* $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ *is the* RS-*mesure space and* E *is a bounded measurable set in* $\boldsymbol{R}^d$ *and* K *is an arbitrary closed interval which include* E. *Let* $\chi_E(x)$ *be the defining function of* E. *We assume that* $f(x)$ *is a bounded integrable function on* E *and we put*

$$f^*(x) = \begin{cases} f(x), & (x \in E), \\ 0, & (x \in K \backslash E) \end{cases} \tag{3.9}$$

and we denote this as $f^*(x) = f(x)\chi_E(x)$ *formally. Then we have the equality*

$$\int_E f(x)d\mu = \int_K f(x)\chi_E(x)d\mu. \tag{3.10}$$

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