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Transfinite Version of Welter's Game

By

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Abstract

We study the transfinite version of Welter's Game, a combinatorial game played on a belt divided into squares numbered with general ordinal numbers. In particular, we give a solution for the game, based on those of the transfinite version of Nim and the original version of Welter's Game using Cantor normal form.

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Introduction

We assume that the reader is familiar with basic terminology on combinatorial game theory, in particular about impartial games, for example, $\mathcal{N}$-position, $\mathcal{P}$-position, nim-sum, and $\mathcal{G}$-values (see [2], [7]).

Welter's Game is an impartial game investigated by Welter in 1954 [9]. Since it was also investigated by Mikio Sato, it is often called Sato's Maya Game in Japan. The rules of Welter's games are as follows:

- It is played with several coins placed on a belt divided into squares numbered with the nonnegative integers $0, 1, 2, \ldots$ from the left as shown in Fig. 1.
- The legal move is to move any one coin from its present square to any unoccupied square with a smaller number.
- The game terminates when a player is unable to move any coin, namely, the coins are jammed in the squares with the smallest numbers as shown in Fig. 2.

We express the binary nim-sum by symbol $\oplus$, and finitely many summation by $\bigoplus$.

Definition 0.1 (Welter function)**.** Let $(a_1, \ldots, a_n)$ be a Welter's Game position. Then we define the value $[a_1|\cdots|a_n]$ of Welter function at $(a_1, \ldots, a_n)$ as follows:

$$[a_1|\cdots|a_n] = a_1 \oplus \cdots \oplus a_n \oplus \bigoplus_{1 \leq i < j \leq n} (a_i \mid a_j),$$

where $(a_i \mid a_j) = (a_i \oplus a_j) \oplus (a_i \oplus a_j - 1)$.

The $\mathcal{G}$-value of the position in Welter's Game is known to be computed by Welter function.

Theorem 0.2 (Welter's Theorem [9])**.** The value of Welter function at each position gives its $\mathcal{G}$-value. Namely,

$$\mathcal{G}(a_1, \ldots, a_n) = [a_1|\ldots|a_n].$$

The transfinite version of Welter's Game was introduced in [6], and the transfinite version of Nim is written in [2] for example. Kayada gave closed-form expressions for the $\mathcal{G}$-values of the transfinite versions of Nim and Welter's Game. He claimed they are correct and announced that he would publish the proof for the transfinite version of Welter's Game [5], but his proof has not been published yet.

In this paper, we give different expressions of the $\mathcal{G}$-values using the Cantor normal form of ordinal numbers and give a new proofs for them.

1. Transfinite Game

We denote by $\mathbb{Z}$ the set of all integers and by $\mathbb{N}_0$ the set of all nonnegative integers.

Let us denote by $\mathcal{ON}$ the class of all ordinal numbers. Later we see that the nim-sum operation can be extended naturally on $\mathcal{ON}$.

The following is known about general ordinal numbers.

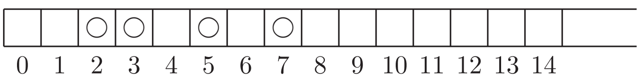

Figure 1: Welter's Game

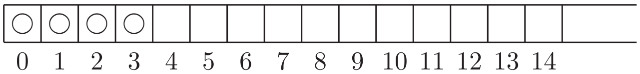

Figure 2: End position of Welter's Game

Theorem 1.1 (Cantor Normal Form theorem [4])**.** Every $\alpha \in \mathcal{ON}$ $(\alpha > 0)$ can be expressed as

$$\alpha = \omega^{\gamma_k} \cdot m_k + \cdots + \omega^{\gamma_1} \cdot m_1 + \omega^{\gamma_0} \cdot m_0,$$

where ω is the lowest transfinite ordinal number and k is a nonnegative integer, $m_0, \ldots, m_k \in \mathbb{N}_0 \setminus \{0\}$, and $\alpha \geq \gamma_k > \cdots > \gamma_1 > \gamma_0 \geq 0$.

Let $\alpha_1, \ldots, \alpha_n$ be ordinal numbers. Then, each α_i, $i = 1, \ldots, n$ is expressed by using finitely many common powers $\gamma_0, \ldots, \gamma_k$ as:

$$\alpha_i = \omega^{\gamma_k} \cdot m_{ik} + \cdots + \omega^{\gamma_1} \cdot m_{i1} + \omega^{\gamma_0} \cdot m_{i0},$$

where $m_{ik} \in \mathbb{N}_0$.

Next, we will define the minimal excluded number of a class of ordinal numbers and the $\mathcal{G}$-value of a position in general Transfinite Game.

Definition 1.2 (minimal excluded number)**.** Let T be a proper subclass of $\mathcal{ON}$. Then mex T is defined to be the least ordinal number not contained in T, namely

$$\operatorname{mex} T = \min(\mathcal{ON} \setminus T).$$

Definition 1.3. Let G and G' be game positions. The notation $G \to G'$ means that G' can be reached from G by a single move.

Definition 1.4 ($\mathcal{G}$-value)**.** Let G and G' be game positions. The value $\mathcal{G}(G)$ called the $\mathcal{G}$-value of G is defined as follows:

$$\mathcal{G}(G) = \operatorname{mex}\{\mathcal{G}(G') \mid G \to G'\}.$$

Theorem 1.5. Let G be a game position. Then

$$\begin{cases} \mathcal{G}(G) \neq 0 \text{ if and only if } G \text{ is an } \mathcal{N}\text{-position}, \\ \mathcal{G}(G) = 0 \text{ if and only if } G \text{ is a } \mathcal{P}\text{-position}. \end{cases}$$

2. Main Results

2.1. Transfinite Nim

We extend Nim into its transfinite version (Transfinite Nim) by allowing the size of the heaps of tokens to be a general ordinal number. The legal moves are to replace an arbitrary ordinal number α by a smaller ordinal number β.

Definition 2.1. For ordinal numbers $\alpha_1, \ldots, \alpha_n \in \mathcal{ON}$, we define their nim-sum as follows:

$$\alpha_1 \oplus \cdots \oplus \alpha_n = \sum_k \omega^{\gamma_k}(m_{1k} \oplus \cdots \oplus m_{nk}).$$

We give a proof for the $\mathcal{G}$-value of the Transfinite Nim using this form.

Theorem 2.2. For Transfinite Nim position $(\alpha_1, \ldots, \alpha_n) \subseteq \mathcal{ON}^n$, we have the following:

$$\mathcal{G}(\alpha_1, \ldots, \alpha_n) = \alpha_1 \oplus \cdots \oplus \alpha_n.$$

Proof. The proof is by induction. Let $\alpha_1 \oplus \cdots \oplus \alpha_n = \alpha$ $(\alpha \in \mathcal{ON})$. We have to show that, for each β $(< \alpha)$, there exists a position with $\mathcal{G}$-value β reached by a single move from $(\alpha_1, \ldots, \alpha_n)$.

Let $(\alpha_1, \ldots, \alpha_n) \to (\beta_1, \ldots, \beta_n)$, by induction hypothesis we have

$$\mathcal{G}(\beta_1, \ldots, \beta_n) = \beta_1 \oplus \cdots \oplus \beta_n.$$

If $\alpha = 0$, no ordinal β $(\beta < \alpha)$ exists. We can assume $\alpha > 0$.

We can write α and β as

$$\begin{aligned} \alpha &= \omega^{\gamma_k} \cdot a_k + \cdots + \omega^{\gamma_k} \cdot a_1 + a_0 \\ \beta &= \omega^{\gamma_k} \cdot b_k + \cdots + \omega^{\gamma_k} \cdot b_1 + b_0, \end{aligned}$$

where $a_0, \ldots, a_k, b_0, \ldots, b_k \in \mathbb{N}_0$. By definition,

$$a_s = m_{1s} \oplus \cdots \oplus m_{ns}, \text{ for } s = 1, \ldots, k.$$

Since $\alpha > \beta$, there exsists s such that

$$a_s > b_s,\ a_t = b_t \text{ for all } t\ (< s).$$

As in the strategy of original Nim, since $a_s > b_s$, there is an index i such that

$$m_{is} > m_{is} \oplus a_s \oplus b_s.$$

We define

$$m'_{it} = m_{it} \oplus a_s \oplus b_s \text{ for all } t\ (\leq s)$$

and

$$\begin{aligned} \alpha'_i = \omega^{\gamma_k} \cdot m_{ik} + \cdots \omega^{\gamma_s + 1} \cdot m_{is+1} + \omega^{\gamma_s} \cdot m'_{is} \\ + \omega^{\gamma_s - 1} \cdot m'_{is-1} + \cdots + \omega^{\gamma_0} \cdot m'_{i0}, \end{aligned}$$

where $m_{is} \oplus a_s \oplus b_s = m'_{is}$.

If we put $\alpha'_i = \beta_i$, $\alpha_j = \beta_j$ $(j \neq i)$. Then, $\alpha_i > \beta_i$ and we have

$$\beta_1 \oplus \cdots \beta_{i-1} \oplus \beta_i \oplus \beta_{i+1} \oplus \cdots \beta_n = \beta$$

Therefore, for each β $(< \alpha)$, there is a position $(\beta_1, \ldots, \beta_n)$ reached by a single move from $(\alpha_1, \ldots, \alpha_n)$. $\square$

Example 2.3. In the case of position $(1, \omega \cdot 2 + 4, \omega^2 \cdot 3 + 9, \omega^2 \cdot 2 + \omega \cdot 4 + 16, \omega^2 + \omega \cdot 5 + 25)$:

Let us calculate the value of $\alpha_1 \oplus \alpha_2 \oplus \alpha_3 \oplus \alpha_4 \oplus \alpha_5$.

We get

$$\begin{aligned}
\alpha_1 &= \omega^{\beta_2} \cdot m_{12} + \omega^{\beta_1} \cdot m_{11} + m_{10} = \omega^2 \cdot 0 + \omega \cdot 0 + 1 \\
\alpha_2 &= \omega^{\beta_2} \cdot m_{22} + \omega^{\beta_1} \cdot m_{21} + m_{20} = \omega^2 \cdot 0 + \omega \cdot 2 + 4 \\
\alpha_3 &= \omega^{\beta_2} \cdot m_{32} + \omega^{\beta_1} \cdot m_{31} + m_{30} = \omega^2 \cdot 3 + \omega \cdot 0 + 9 \\
\alpha_4 &= \omega^{\beta_2} \cdot m_{42} + \omega^{\beta_1} \cdot m_{41} + m_{40} = \omega^2 \cdot 2 + \omega \cdot 4 + 16 \\
\alpha_5 &= \omega^{\beta_2} \cdot m_{52} + \omega^{\beta_1} \cdot m_{51} + m_{50} = \omega^2 \cdot 1 + \omega \cdot 5 + 25.
\end{aligned}$$

So, we have

$$\begin{aligned}
m_{12} \oplus m_{22} \oplus m_{32} \oplus m_{42} \oplus m_{52} &= 0 \oplus 0 \oplus 3 \oplus 2 \oplus 1 \\
&= 0 \\
m_{11} \oplus m_{21} \oplus m_{31} \oplus m_{41} \oplus m_{51} &= 0 \oplus 2 \oplus 0 \oplus 4 \oplus 5 \\
&= 3 \\
m_{10} \oplus m_{20} \oplus m_{30} \oplus m_{40} \oplus m_{50} &= 1 \oplus 4 \oplus 9 \oplus 16 \oplus 25 \\
&= 5.
\end{aligned}$$

Thus, by the definition of nim-sum in general ordinal number

$$\alpha_1 \oplus \alpha_2 \oplus \alpha_3 \oplus \alpha_4 \oplus \alpha_5 = \omega \cdot 3 + 5.$$

Therefore, this position is an $\mathcal{N}$-position, and the legal good move is $\omega \cdot 2 + 4 \to \omega + 1$.

2.2. Transfinite Welter's Game

In transfinite version (Transfinite Welter's Game), the size of the belt of Welter's Game is extended into general ordinal numbers, but played with finite number of coins. The legal moves are to move one coin toward the left (jumping is allowed), and one cannot place two or more coins on the same square as in the original Welter's Game (see Fig. 3). We will define Welter function of a position of Transfinite Welter's Game.

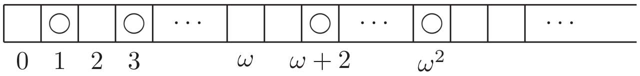

Figure 3: Transfinite Version of Welter's Game

Definition 2.4. Let $\alpha_1, \dots, \alpha_n \in \mathcal{ON}$. Each α_i can be expressed as $\alpha_i = \omega \cdot \lambda_i + m_i$, where $\lambda_i \in \mathcal{ON}$ and $m_i \in \mathbb{N}_0$. Welter function in general ordinal numbers is defined as follows:

$$[\alpha_1 | \cdots | \alpha_n] = \omega \cdot (\lambda_1 \oplus \cdots \oplus \lambda_n) + \bigoplus_{\lambda \in \mathcal{ON}} [S_\lambda],$$

where $[S_\lambda]$ is Welter function, and $S_\lambda = \{m_n \mid \lambda_n = \lambda\}$.

The following is our main result.

Theorem 2.5. Let $\alpha_1, \dots, \alpha_n \in \mathcal{ON}$. The $\mathcal{G}$-value of general position $(\alpha_1, \dots, \alpha_n)$ in Transfinite Welter's Game is equal to its Welter function. Namely,

$$\mathcal{G}(\alpha_1, \dots, \alpha_n) = [\alpha_1 | \cdots | \alpha_n].$$

Proof. Let $[\alpha_1 | \cdots | \alpha_n] = \alpha$. We have to show that, for each β $(< \alpha)$, there exists a position with $\mathcal{G}$-value β reached by a single move from $(\alpha_1, \dots, \alpha_n)$.

Let $(\alpha_1, \dots, \alpha_n) \to (\beta_1, \dots, \beta_n)$. Then by the assumption of induction we have

$$\mathcal{G}(\beta_1, \dots, \beta_n) = [\beta_1 | \cdots | \beta_n].$$

If $\alpha = 0$, there exist no β $(< \alpha)$. We can assume $\alpha > 0$ and

$$\alpha = \omega \cdot \lambda + a_0 \text{ and } \beta = \omega \cdot \lambda' + b_0,$$

where λ, $\lambda' \in \mathcal{ON}$, a_0, $b_0 \in \mathbb{N}_0$. Since $\alpha > \beta$, we have

$$(\lambda > \lambda') \text{ or } (\lambda = \lambda' \text{ and } a_0 > b_0).$$

In the latter case, since $a_0 = \bigoplus_{\lambda \in \mathcal{ON}} [S_\lambda] > b_0$, from the theory of Nim ([1], [3], [8]) there exists some λ_0 and nonnegative integer c_0 $(< [S_{\lambda_0}])$ such that

$$a_0 \oplus [S_{\lambda_0}] \oplus c_0 = b_0.$$

Next since $[S_{\lambda_0}] > c_0$, from the theory of Welter function ([2]), there is an index i and m'_i $(< m_i)$ such that $m_i \in S_{\lambda_0}$ and $[S'_{\lambda_0}] = c_0$, where S'_{λ_0} is the set obtained from S_{λ_0} by replacing m_i with m'_i. Thus, the move from $\alpha_i = \omega \cdot \lambda_i + m_i$ to $\alpha'_i = \omega \cdot \lambda_i + m'_i$ changes its $\mathcal{G}$-value from $\alpha = \omega \cdot \lambda + a_0$ to $\beta = \omega \cdot \lambda + b_0$.

In the former case, as in Transfinite Nim, there is an index i and λ'_i $(< \lambda)$ such that $(\lambda_1, \ldots, \lambda_{i-1}, \lambda'_i, \lambda_{i+1}, \ldots, \lambda_n)$ has $\mathcal{G}$-value λ' and we can adjust the finite part of α_i so that the resulting Welter function to be β.

Therefore, for each β $(< \alpha)$, there is a position reached by a single move from $(\alpha_1, \ldots, \alpha_n)$ and its $\mathcal{G}$-value is β. $\square$

Corollary 2.6. A position in Transfinite Welter's Game is a $\mathcal{P}$-position if and only if it satisfies the following conditions:

$$\begin{cases} \omega \cdot (\lambda_1 \oplus \cdots \oplus \lambda_n) = 0 \\ \displaystyle\bigoplus_{\lambda \in \mathcal{ON}} [S_\lambda] = 0. \end{cases}$$

By this corollary, we can easily calculate a winning move in Transfinite Welter's Game by its Welter function.

Example 2.7. In the case of position $(1, \omega{\cdot}2{+}4, \omega{\cdot}2{+}9, \omega^2{+}\omega{\cdot}4{+}16, \omega^2{+}\omega{\cdot}5{+}25)$:

Let us calculate the value of $[\alpha_1|\alpha_2|\alpha_3|\alpha_4|\alpha_5]$. We get

$$\begin{aligned} \alpha_1 &= \omega^{\beta_2} \cdot m_{12} + \omega^{\beta_1} \cdot m_{11} + m_{10} = \omega^2 \cdot 0 + \omega \cdot 0 + 1 \\ \alpha_2 &= \omega^{\beta_2} \cdot m_{22} + \omega^{\beta_1} \cdot m_{21} + m_{20} = \omega^2 \cdot 0 + \omega \cdot 2 + 4 \\ \alpha_3 &= \omega^{\beta_2} \cdot m_{32} + \omega^{\beta_1} \cdot m_{31} + m_{30} = \omega^2 \cdot 0 + \omega \cdot 2 + 9 \\ \alpha_4 &= \omega^{\beta_2} \cdot m_{42} + \omega^{\beta_1} \cdot m_{41} + m_{40} = \omega^2 \cdot 1 + \omega \cdot 4 + 16 \\ \alpha_5 &= \omega^{\beta_2} \cdot m_{52} + \omega^{\beta_1} \cdot m_{51} + m_{50} = \omega^2 \cdot 1 + \omega \cdot 5 + 25. \end{aligned}$$

So, we have

$$\begin{aligned} m_{12} \oplus m_{22} \oplus m_{32} \oplus m_{42} \oplus m_{52} &= 0 \oplus 0 \oplus 0 \oplus 1 \oplus 1 \\ &= 0 \\ m_{11} \oplus m_{21} \oplus m_{31} \oplus m_{41} \oplus m_{51} &= 0 \oplus 2 \oplus 2 \oplus 4 \oplus 5 \\ &= 1 \\ [m_{10}] \oplus [m_{20} \mid m_{30}] \oplus [m_{40}] \oplus [m_{50}] &= [1] \oplus [4 \mid 9] \oplus [16] \oplus [25] \\ &= 1 \oplus (4 \oplus 9 - 1) \oplus 16 \oplus 25 \\ &= 4. \end{aligned}$$

Therefore, by the definition of Welter function for general ordinal number

$$[\alpha_1|\alpha_2|\alpha_3|\alpha_4|\alpha_5] = \omega + 4.$$

Since, this shows that we are in an $\mathcal{N}$-position, we will calculate a winning move.

First, we choose a move that satisfies the first condition of Corollary 2.6. Clearly we should not make a move that will change the coefficient of $\omega^{\beta_2} = \omega^2$. So we will choose a move that will change the coefficient of $\omega^{\beta_1} = \omega^1$ to be 0. The same strategy in Transfinite Nim, shows that

$$(2 \oplus 2 \oplus 4 \oplus 5) \oplus 1 = 1 \oplus 1 = 0.$$

Thus, the only legal move is $5 \to 5 \oplus 1 = 4$. So, our good move is in $\omega \cdot 5 + 25$. Then, in such moves, we will search for a move that satisfy the second condition. It is obtained from the knowledge of Welter function.

The finite part should satisfy

$$1 \oplus [4 \mid 9] \oplus [x \mid 16] = 0.$$

So we have

$$x = 30.$$

Therefore, the only good move is $\omega \cdot 5 + 25 \to \omega \cdot 4 + 30$.

In fact,

$$\begin{aligned}
m_{12} \oplus m_{22} \oplus m_{32} \oplus m_{42} \oplus m_{52} &= 0 \oplus 0 \oplus 0 \oplus 1 \oplus 1 \\
&= 0 \\
m_{11} \oplus m_{21} \oplus m_{31} \oplus m_{41} \oplus m_{51} &= 0 \oplus 2 \oplus 2 \oplus 4 \oplus 4 \\
&= 0 \\
[m_{10}] \oplus [m_{20} \mid m_{30}] \oplus [m_{40}] \oplus [m_{50}] &= [1] \oplus [4 \mid 9] \oplus [30 \mid 16] \\
&= 1 \oplus (4 \oplus 9 - 1) \oplus (30 \oplus 16 - 1) \\
&= 1 \oplus 12 \oplus 13 \\
&= 0.
\end{aligned}$$

Thus, this position is a $\mathcal{P}$-position.

3. Acknowledgements

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