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Delete Nim

By

Tomoaki Abuku

National Institute of Informatics,
2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430 Japan.
e-mail address : buku3416@gmail.com

and

Koki Suetsugu

National Institute of Informatics,
2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430 Japan.
e-mail address : koki.suetsugu@gmail.com

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Abstract

We study a game (called Delete Nim) that requires the OR operation to calculate the $\mathcal{G}$-values of its positions. Also, the concept called 2-adic valuation, which is described in number theory, is used. This is very rare in the analysis of impartial games, while the XOR operation is commonly used for calculations of the $\mathcal{G}$-values.

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Introduction

0.1. Combinatorial Games

Combinatorial games are 2-player games with neither chance elements nor hidden information (for the details, see [1, 2, 4, 7]). Nim is the most famous combinatorial game and was studied by Bouton in 1902 [3]. The rules of Nim are as follows:

- The game is played with several heaps of stones.
- Two players move alternately.
- On their turn, the players choose a heap and remove an arbitrary number of stones.
- The player who removes the last stone wins.

It is natural to define the winner of a combinatorial game as the player who makes the last move, and we call this normal convention. Furthermore, a game where both players always have the same set of possible moves is called impartial. In this study, we investigate impartial games that played under the normal convention and end in a finite number of moves.

When we say "a player has a winning strategy," it means that they can win the game no matter how the opponent moves.

Definition 0.1. When the next (resp. previous) player has a winning strategy, the position is called an $\mathcal{N}$-position (resp. a $\mathcal{P}$-position).

The set of positions can be partitioned into two sets corresponding to the $\mathcal{N}$-positions and $\mathcal{P}$-positions.

Definition 0.2. The addition of numbers in their binary form without carrying is called the XOR operation (or Nim-sum). We denote the XOR operation by $\oplus$. Thus, the result of the XOR operation of nonnegative integers $m_1, \dots, m_n$ is written as

$$m_1 \oplus \cdots \oplus m_n.$$

The XOR operation is commonly used to determine whether a Nim position is an $\mathcal{N}$-position or a $\mathcal{P}$-position.

Theorem 0.3 (Bouton [3])**.** The Nim position $(n_1, n_2, \dots, n_k)$ is a $\mathcal{P}$-position if and only if $n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$.

Example 0.4. $3 \oplus 5 \oplus 6 = 11_2 \oplus 101_2 \oplus 110_2 = 0$; therefore, the Nim position $(3, 5, 6)$ is a $\mathcal{P}$-position.

Example 0.5. $2 \oplus 4 \oplus 7 = 10_2 \oplus 100_2 \oplus 111_2 = 1_2 = 1$; therefore, the Nim position $(2, 4, 7)$ is an $\mathcal{N}$-position.

0.2. $\mathcal{G}$-values for Impartial Games

Sprague [8] and Grundy [5] extended Bouton's theorem for general impartial games played under the normal convention. Let us denote the set of all non-negative integers by $\mathbb{N}$.

Definition 0.6. For any proper subset S of $\mathbb{N}$, we define the minimal excluded function $\mathrm{mex}(S)$ as follows:

$$\mathrm{mex}(S) = \min(\mathbb{N} \setminus S).$$

Definition 0.7. For any game position G, we define the $\mathcal{G}$-value function $\mathcal{G}(G)$ as follows:

$$\mathcal{G}(G) = \mathrm{mex}(\{\mathcal{G}(G') \mid G \to G'\}),$$

where $G \to G'$ means that G' can be reached from G by a single move.

Theorem 0.8 (Sprague [8] and Grundy [5])**.** For any game position G, G is a $\mathcal{P}$-position if and only if $\mathcal{G}(G) = 0$.

Therefore, by calculating the $\mathcal{G}$-value, it is possible to determine which player has a winning strategy.

The $\mathcal{G}$-value is also useful for analysis of the disjunctive sum of games. For any two positions of impartial games G and H, we define the disjunctive sum of G and H (written as $G + H$) as follows. Each player must make a move to either G or H (but not both) on their turn.

Theorem 0.9 (Sprague [8] and Grundy [5])**.** Let G and H be two game positions. Then

$$\mathcal{G}(G + H) = \mathcal{G}(G) \oplus \mathcal{G}(H).$$

Therefore, researchers have been interested in $\mathcal{G}$-values of games and a few of the early results have shown various structures of $\mathcal{G}$-values in a few specific games.

When we tried to analyze the game called "Hey, That's My Fish!" [6], we simplified it (changed it to an impartial, each position has one piece, and the board changed to $1 \times n$, etc.). Then, we invented the following Delete Nim.

0.3. Rules of Delete Nim

The rules of Delete Nim are as follows:

- There are two heaps of tokens.
- Two players move alternately.

- The player selects a non-empty heap and deletes the other heap and removes 1 token from the selected heap and splits the heap into two (possibly empty heaps).
- The player who can not take stones loses.

Example 0.10. In the case of $(11, 9)$.
$(11, \underline{9}) \to (\underline{5}, 3) \to (\underline{4}, 0) \to (\underline{2}, 1) \to (\underline{1}, 0) \to (0, 0)$.

Here $(\underline{a}, b)$ denotes the player selects the first heap a, and $(a, \underline{b})$ denotes the player selects the second heap b, respectively.

0.4. OR operation

The OR operation is required to compute the $\mathcal{G}$-value of the position of Delete Nim.

Definition 0.11. We denote by $\vee$ the usual OR operation of two numbers in binary notation.

Example 0.12. $3 \vee 5 = 11_2 \vee 101_2 = 111_2 = 7$.

Example 0.13. $9 \vee 10 = 1001_2 \vee 1010_2 = 1011_2 = 11$.

1. Main Results

Theorem 1.1. We denote the position of Delete Nim with two heaps of x tokens and y tokens by (x, y). Then,

$$\mathcal{G}((x, y)) \quad = \quad v_2((x \vee y) + 1),$$

where $v_p(n)$ is the p-adic valuation of n; that is,

$$v_p(n) = \begin{cases} \max\{l \in \mathbb{N} : p^l \mid n\} & (n \neq 0) \\ \infty & (n = 0). \end{cases}$$

Proof. Let $x = \sum_i 2^i x_i$, $y = \sum_i 2^i y_i$ $(x_i, y_i \in \{0, 1\})$ and $h = v_2((x \vee y) + 1)$.

First, we show that (x, y) has no next position (x', y') such that $h = v_2((x' \vee y') + 1)$. Note that $x' + y' = x - 1$ or $x' + y' = y - 1$.

If $h = 0$, then x and y are even. Therefore, $x' + y'$ is an odd number and $v_2((x' \vee y') + 1) \neq 0$, which is a contradiction.

Let $x' = \sum_i 2^i x'_i$, $y' = \sum_i 2^i y'_i$ $(x'_i, y'_i \in \{0, 1\})$. If $h > 0$, then $x'_h = y'_h = 0$ and for any $k < h$, $x'_k = 1$ or $y'_k = 1$. Therefore, $2^h - 1 \leq ((x' + y') \bmod 2^{h+1}) \leq 2^{h+1} - 2$, and thus, $2^h \leq ((x' + y' + 1) \bmod 2^{h+1}) \leq 2^{h+1} - 1$. Then, $x'_h = 1$ or $y'_h = 1$, which is a contradiction.

Next, we show that for any $h' < h$, (x, y) has a next position such that $h' = v_2((x' \vee y') + 1)$. Since $h = v_2((x \vee y) + 1)$, without loss of generality, assume that $x_{h'} = 1$. Let $x' = x - 2^{h'}$ and $y' = 2^{h'} - 1$. Evidently, $x'_{h'} = 0$, $y'_{h'} = 0$, $y'_k = 1$ $(k < h')$, and $x' + y' = x - 1$. Therefore, (x', y') is a next position of (x, y) and $h' = v_2((x' \vee y') + 1)$. □

Example 1.2. In the case of $(9, 10)$.

$$\begin{aligned}\mathcal{G}((9,10)) &= v_2((9 \vee 10) + 1)\\ &= v_2(12)\\ &= 2.\end{aligned}$$

Example 1.3. In the case of $(8, 12)$.

$$\begin{aligned}\mathcal{G}((8,12)) &= v_2((8 \vee 12) + 1)\\ &= v_2(13)\\ &= 0.\end{aligned}$$

A game similar to Delete Nim was introduced in [9]. We call this game Variant of Delete Nim (VDN). The rules of VDN are as follows:

- There are two (non-empty) heaps of tokens.
- Two players move alternately.
- The player selects one of the heaps and deletes it, and splits the other heap into two (non-empty) heaps.
- The player who can not take stones loses.

Example 1.4. In the case of $(8, 5)$.
$(\underline{8}, 5) \to (\underline{7}, 1) \to (4, \underline{3}) \to (\underline{2}, 1) \to (1, 1)$.

The $\mathcal{N}$-positions and $\mathcal{P}$-positions of VDN have been already shown [9]. However, the $\mathcal{G}$-values of the positions of the game have not been discussed.

Theorem 1.5 ([9])**.** Let G be a position of VDN.

At least one of the heaps has an even numbers of tokens $\iff$ G is an $\mathcal{N}$-position.
Both heaps have an odd number of tokens $\iff$ G is a $\mathcal{P}$-position.

VDN can be shown to be isomorphic to Delete Nim, in the following sense.

Definition 1.6. Let G and H be game positions. We say that G is isomorphic to H, if G and H have the same game tree.

Lemma 1.7. Let F be the function denoted by

$$F(x,y)=(x-1,y-1).$$

Then, F is an isomorphism from the set of all positions of VDN to that of Delete Nim. Namely,

$$(x,y)\to(x',y')\iff F(x,y)\to F(x',y').$$

Proof. This is easily proven, so we omit the proof. □

By Theorem 1.1 and Lemma 1.7, we can compute the $\mathcal{G}$-value of position (x,y) of VDN.

Theorem 1.8. Let (x,y) be a position of VDN, then we have

$$\mathcal{G}((x,y))=v_2(((x-1)\vee(y-1))+1).$$

2. Conclusion

In this study, we introduced a game for which one needs to use the OR operation and 2-adic valuation $v_2(n)$ to compute the $\mathcal{G}$-value of a position. This is very rare in the analysis of impartial games. Therefore, we hope that our results will expand potential analysis strategies. There may be games that require other operations such as the AND operation to compute the $\mathcal{G}$-values; however, so far, only trivial ones have been discovered. In the future, we would like to consider non-trivial games that require other operations. We are also interested in some extensions of Delete Nim, such as a similar game with more than two heaps. However, this problem is difficult and so far we know nothing about it.

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