

Paper

# Calculation method for unstable periodic points in two-to-one maps using symbolic dynamical system

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**Abstract:** In this study, we have focused on the two-to-one maps and developed the numerical method to calculate the unstable periodic points (UPPs), based on the theory of the symbolic dynamical system. The core technique of the method is the definition of a non-deterministic map  $G$ . From the experimental result of three typical maps: logistic map, tent map, and Bernoulli map, we have confirmed the proposed method works very well within the defined errors. Our method has the following advantages: the method converges rapidly as the period of the target UPP is larger; we can choose the target UPP regardless of its cause (any bifurcation is not a matter); we can find the UPPs that are always unstable in the given parameter range. The convergence of the method is guaranteed by two standpoints: the corresponding symbolic dynamical system, and the asymptotic stability of UPP of  $G$ . Hereby, the error of the convergence is scalable according to the numeric precision of the software.

**Key Words:** two-to-one map, unstable periodic point, symbolic dynamical system

## 1. Introduction

There are many pieces of research focusing on chaos in the nonlinear dynamical system. As an achievement of such research, Smale [1] has proposed chaos includes infinite unstable periodic points (UPPs). The existence of UPPs affects not only the behavior of chaos but also the steady states of the system. For those reasons, finding UPPs plays a very important role in the analysis of nonlinear systems. As the conventional method, Kawakami [2] has suggested the method, based on Newton's method, to calculate the location and stability of a UPP. However, such methods require an initial point coordinate near the corresponding UPP, which is often unknown in many cases, e.g., when the UPP has a larger period.

On the other hand, symbolic dynamics have contributed to analyzing the qualitative characteristics

of chaos [3] [4]. We can interpret the dynamical system showing chaos with a symbolic system called a full 2-shift and analyze the original system from the standpoint of symbolic dynamics.

In this study, we focus on the two-to-one maps and discover a novel method to calculate their UPPs based on the symbolic dynamical system. The core technique of our method is the definition of a non-deterministic map  $G$  whose inverse is the target two-to-one map.

## 2. System description

### 2.1 Two-to-one map and dynamical system

We consider the one-dimensional two-to-one map, which is the map having single extreme value:

$$F : X \rightarrow S; x \mapsto F(x), \quad \text{such that} \quad X = X_0 \cup X_1, F(X_0) = F(X_1) = S, \quad (1)$$

where  $X_i \subset \mathbb{R}$  and  $S = \{x \in \mathbb{R} \mid x < F(\alpha)\}$ . A two-to-one map is non-invertible, however, we can consider two maps whose inverses construct it:

$$G_0 : S \rightarrow X_0; \quad x \mapsto G_0(x), \quad X_0 = \{x \in X \mid x \leq \alpha\}, \quad (2)$$

$$G_1 : S \rightarrow X_1; \quad x \mapsto G_1(x), \quad X_1 = \{x \in X \mid x > \alpha\}. \quad (3)$$

From these maps, let us alternatively define an inverse of  $F$  as a non-deterministic map  $G$ :

$$G : S \rightarrow X = X_0 \cup X_1; \quad x \mapsto G(x) = G_m(x), \quad m = 0, 1. \quad (4)$$

For some examples, logistic map and tent map are typical two-to-one maps, as shown in Fig. 1.

Focusing on the discrete-time dynamical system  $x_{n+1} = F(x_n), n \in \mathbb{Z}$ , we consider the unstable periodic point (UPP) of this system. We write the trajectory of this system through  $x_0$  as

$$\{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}. \quad (5)$$

This trajectory is deterministic for the time-forward direction:  $x_1, x_2$ , and further, but is not for the time-backward direction:  $x_{-1}, x_{-2}$ , and so on, because  $F$  is deterministic but  $G$  is not. This definition allows us to choose the backward image of  $x^*$  by choosing  $G_0$  or  $G_1$ .

An  $\ell$ -periodic point of  $F$  is the point  $x^*$  satisfying  $x^* = F^\ell(x^*)$  with  $x^* \neq F^k(x^*)$  for  $1 \leq k < \ell$ , and  $\mu = dF^\ell/dx(x^*)$  determines the asymptotic stability of  $x^*$ , i.e.  $x^*$  is stable if  $|\mu| < 1$  and unstable otherwise. This  $x^*$  of  $F^\ell$  is also the periodic point of  $G^\ell$  with the proper choice of each  $m$ . Let  $\nu = dG^\ell/dx(x^*)$ , which determines the stability of  $x^*$  from the standpoint of  $G$ . As a trivial fact of the algebraic theory,  $\nu = \mu^{-1}$  since  $G^\ell$  is invertible regardless of the choice of  $m$ .

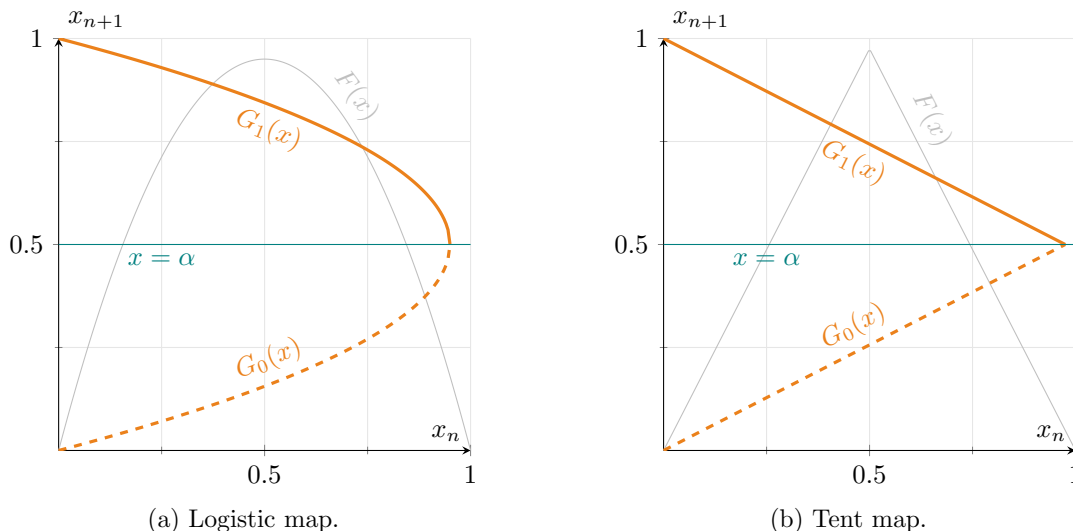


Fig. 1: Example of two-to-one maps  $F(x)$  and corresponding  $G_0(x)$  and  $G_1(x)$ .

## 2.2 Symbolic dynamical system

Let  $\mathcal{A}^{\mathbb{Z}}$  be a sequence space defined by  $\mathcal{A}^{\mathbb{Z}} = \{s = (s_i)_{i \in \mathbb{Z}} \mid s_i \in \mathcal{A}, \forall i \in \mathbb{Z}\}$ , and  $\sigma$  be a shift map  $\sigma : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ ;  $(\sigma s)_i = s_{i+1}$ , then the pair of  $\mathcal{A}^{\mathbb{Z}}$  and  $\sigma$  defines a symbolic system. We call  $\mathcal{A}^{\mathbb{Z}}$  a full 2-shift if  $\mathcal{A} = \{0, 1\}$ . Richer explanation for the symbolic dynamical system is available on Ref.[5].

In this study, we interpret the dynamics of  $F$  and  $G$  as a symbolic system. Let us assign the symbol  $s_i$  with following rules:  $s_i = 0$  if  $x_i \in X_0$ ;  $s_i = 1$  if  $x_i \in X_1$ . Then, a point  $x_0 \in \mathbb{R}$  of the trajectory (5) corresponds to the bi-infinite symbol sequence  $s(x_0) \in \mathcal{A}^{\mathbb{Z}}$ :  $s(x_0) = \cdots s_{-2}s_{-1}.s_0s_1s_2 \cdots$ , where the point ‘.’ located between  $s_{-1}$  and  $s_0$  divides the symbol sequence into two parts,  $s_i, i < 0$  and  $s_i, i \geq 0$ . In general,  $s(x_n) = \cdots s_{n-2}s_{n-1}.s_ns_{n+1} \cdots$  and this leads to

$$s(F(x_0)) = s(x_1) = \cdots s_{-2}s_{-1}s_0.s_1s_2 \cdots = \sigma s(x_0), \quad (6)$$

$$s(G(x_0)) = s(x_{-1}) = \cdots s_{-2}.s_{-1}s_0s_1s_2 \cdots = \sigma^{-1} s(x_0). \quad (7)$$

Hence,  $F$  and  $G$  on  $\mathbb{R}$  are topologically conjugate to the left and right shift map  $\sigma$  and  $\sigma^{-1}$  on  $\mathcal{A}^{\mathbb{Z}}$ , respectively. From the topologically conjugate property, the  $\ell$ -periodic point  $x^*$  of  $F$  corresponds to the symbol sequence  $s^*$  of  $\sigma$ , which is the sequence that infinitely repeats a basic sequence having the length of  $\ell$ . For example, Tab. I shows the correspondence of them.

Table I: Relationships of the  $\ell$ -periodic points between  $F$  and its corresponding symbolic system.

$\ell$	bi-infinite sequence $s^* = s(x^*)$	basic sequence of $s^*$
1	$\cdots 0000.0000 \cdots, \quad \cdots 1111.1111 \cdots$	0, 1
2	$\cdots 1010.1010 \cdots$	10
3	$\cdots 001001.001001 \cdots, \quad \cdots 011011.011011 \cdots$	001, 011

## 3. Method of analysis

As defined in the previous section, we can arbitrary choose  $G_0$  or  $G_1$  as the inverse map  $G$ . Consequently, we also arbitrary choose the symbols  $s_i$  for the time-backward direction  $i < 0$ . As a simple example, we can get the following symbol sequence if we always choose  $G_0$  as  $G$ :

$$s(x_0) = \underbrace{\cdots 0000}_{\text{arbitrary}}.s_0s_1s_2 \cdots.$$

In this case, the symbol sequence conjugate to  $G^N(x_0)$  is

$$s(G^N(x_0)) = \sigma^{-N}(x_0) = \cdots . \underbrace{0000 \cdots 0000}_{N \text{ symbols}} s_0s_1s_2 \cdots,$$

for an arbitrarily large  $N$ , i.e., the larger  $N$  becomes the longer the symbol ‘0’ continues. This fact naturally implies the trajectory of  $G_0$  asymptotically converges to the 1-periodic point that has the basic sequence of ‘0’ in Tab. I. For another example, when we choose  $G_0$  and  $G_1$  alternatively, the symbol sequence is  $s(x_0) = \cdots 01010101.s_0s_1s_2 \cdots$ , thus, the trajectory of  $G^2 = G_1 \circ G_0$  converges to the 2-periodic point that has the basic sequence of ‘01’. Let us generalize these principles for an arbitrary  $\ell$ -periodic point.

**Step 1.** Choose a basic sequence of an  $\ell$ -periodic point;  
e.g.) 001, 011

**Step 2.** Construct the composed inverse map  $G(x_0)$ ;  
e.g.)  $G^\ell(x_0) = G_0 \circ G_0 \circ G_1(x_0)$ ,  $G^\ell(x_0) = G_0 \circ G_1 \circ G_1(x_0)$

**Step 3.** Iterate  $x_{N+1} = G^\ell(x_N)$  with an arbitrarily large  $N$ ;  
if  $|x_{N+1} - x_N| < \epsilon$  with a desired tolerance  $\epsilon$ ,  $x_N$  is the obtained periodic point.

The trajectory of  $G$  is convergent if  $|\nu| < 1$ . Bringing the formula  $\nu = \mu^{-1}$  in Sec. 2.1,  $|\mu| > 1$  is the required condition for the convergence of this method. This is the condition for the UPP and thus the method goes well with the unstable cases of the periodic point, but not for the stable cases. This limitation is not a critical issue because we can use  $F$  with the proper initial condition  $x_0$  to see the stable cases.

## 4. Result of experiment

We have conducted numerical experiments for three systems: logistic map, tent map, and Bernoulli map. In all experiments, we set  $\epsilon = 1 \times 10^{-15}$  and  $x_0 = 0.7$ . We can choose arbitrary  $\epsilon$  according to the numeric precision of the software since the UPP always has  $|\mu| > 1$  regardless of the precision.

### 4.1 Logistic map

Let  $F(x) = rx(1-x)$  be logistic map where  $r$  is an arbitrary parameter. Figure 2 shows the result of experiments in one-parameter bifurcation diagram. From the figure, we can see the proposing method is valid not only for the UPP after period-doubling bifurcation but also for that after tangent bifurcation (see the period-doubling window of a 3-periodic point). Table II is the numerical results of the obtained points with  $r = 4$ . Conventional methods such as Newton's method take a lot iterations to converge when the period of UPP is larger. However, our method quite quickly converges to the desired point even though the UPP has a great period, as shown in the column  $N$  (counts of numerical iterations) in Tab. II. Moreover, we can find the UPP having quite large period such as  $\ell = 20$  or more. Figure 3 shows the obtained  $x^*$  and the trajectory through it in the return map.

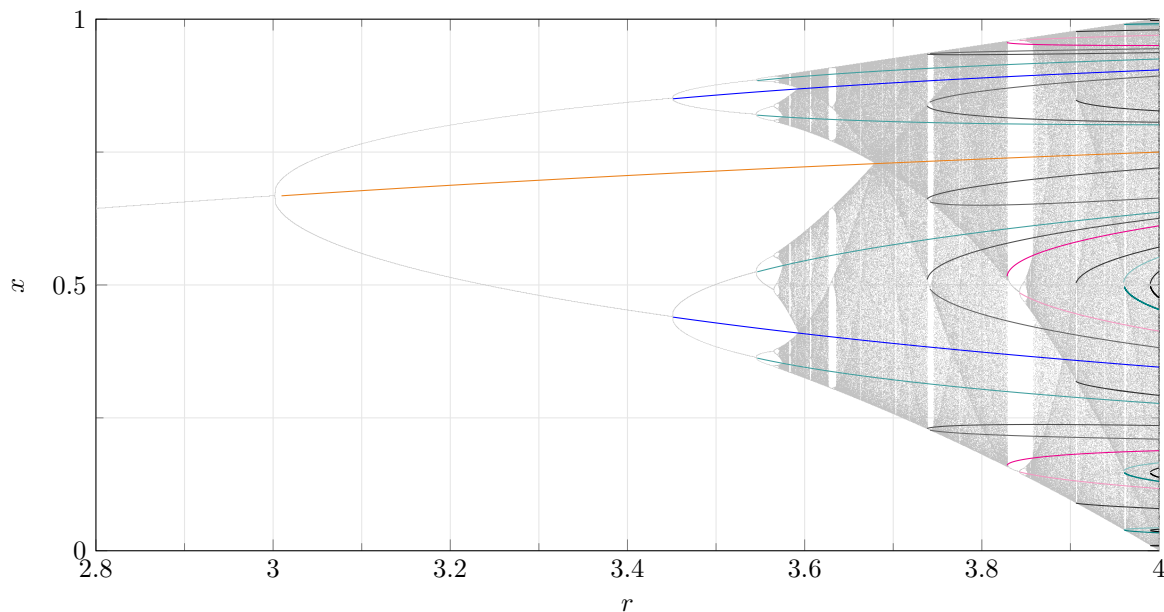


Fig. 2: Bifurcation diagram and UPPs of logistic map with  $r \in [0, 4]$ . Each colored curve represents the UPP with  $\ell = 1$  (orange),  $\ell = 2$  (blue),  $\ell = 3$  (magenta),  $\ell = 4$  (green), and  $\ell = 5$  (black).

Table II: Numerical result for logistic map with  $r = 4$ ,  $x_0 = 0.7$ .

$\ell$	basic sequence	$x^*$	error	$N$	$\nu$
1	0	0.000 000 000 000 000	$8.327 \times 10^{-16}$	27	$2.500 \times 10^{-1}$
	1	0.750 000 000 000 000	$9.992 \times 10^{-16}$	48	$-5.000 \times 10^{-1}$
2	01	0.345 491 502 812 526	$4.441 \times 10^{-16}$	27	$-2.500 \times 10^{-1}$
3	001	0.116 977 778 440 511	$1.665 \times 10^{-16}$	19	$-1.250 \times 10^{-1}$
	011	0.188 255 099 070 633	$1.665 \times 10^{-16}$	19	$1.250 \times 10^{-1}$
20	01101110001110111000	0.201 284 361 839 669	0.000	5	$-9.537 \times 10^{-7}$

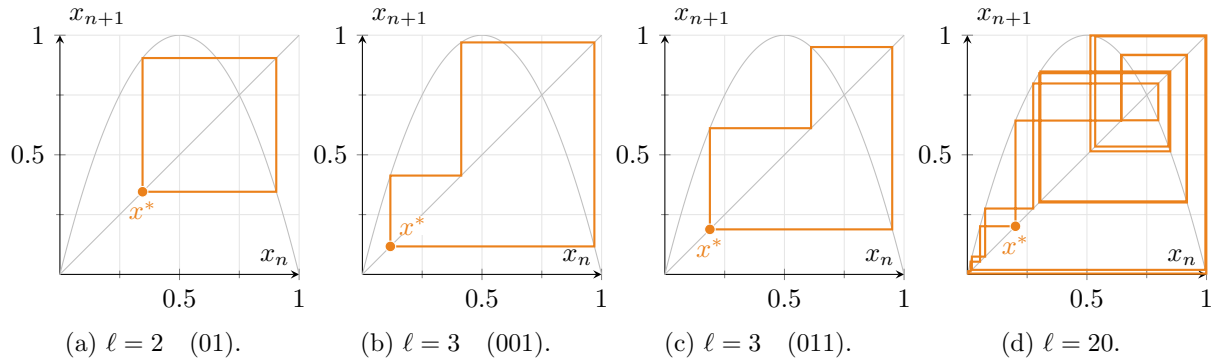


Fig. 3: Return map of UPP  $x^*$  with period- $\ell$  in logistic map with  $r = 4$ .

### 4.2 Tent map

Let  $F(x) = r \min(x, 1 - x)$  be tent map where  $r$  is an arbitrary parameter. The result shown in Fig. 4 indicates that the proposing method is valid for the UPPs caused by border-collision bifurcations[6]. Table III is the numerical results of the obtained UPPs with  $r = 2$ . As the same as logistic map case, our method rapidly converges to the desired UPP. In addition, we can see the correspondence of  $\nu$  comparing with logistic map. This is a natural result because the tent map with  $r = 2$  is the nonlinear transformation of the dyadic transformation[7] and the logistic map with  $r = 4$ . Figure 5 shows the obtained UPP in the return map. Comparing Fig. 3 with 5, we can confirm the conjugacy of the trajectory. As a remarkable fact, we can find the UPPs that always unstable in the given parameter range even though the conventional method cannot trace such a UPP.

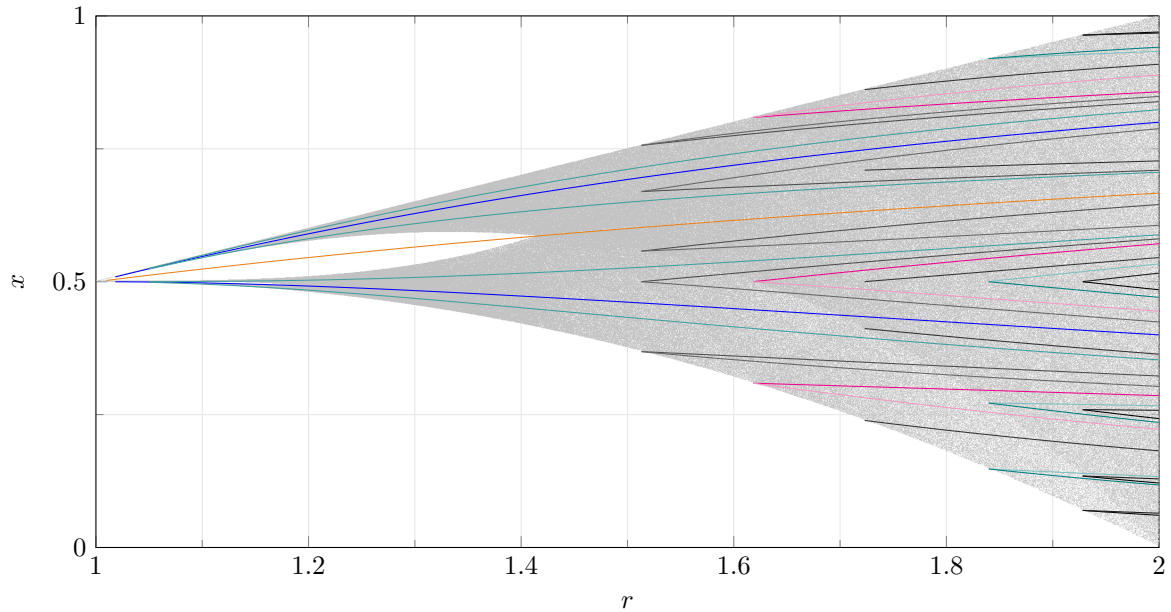


Fig. 4: Bifurcation diagram and UPPs of tent map with  $r \in [0, 2]$  (colors are the same as Fig. 2).

Table III: Numerical result for tent map with  $r = 2$ ,  $x_0 = 0.7$ .

$\ell$	basic sequence	$x^*$	error	$N$	$\nu$
1	0	0.000 000 000 000 000	$2.720 \times 10^{-16}$	19	$1.250 \times 10^{-1}$
	1	0.666 666 666 666 666	$7.772 \times 10^{-16}$	48	$-5.000 \times 10^{-1}$
2	01	0.400 000 000 000 000	$4.441 \times 10^{-16}$	27	$-2.500 \times 10^{-1}$
3	001	0.222 222 222 222 222	$2.220 \times 10^{-16}$	19	$-1.250 \times 10^{-1}$
	011	0.285 714 285 714 286	$1.665 \times 10^{-16}$	19	$1.250 \times 10^{-1}$
20	01101110001110111000	0.296 188 072 025 230	0.000	5	$-9.537 \times 10^{-7}$

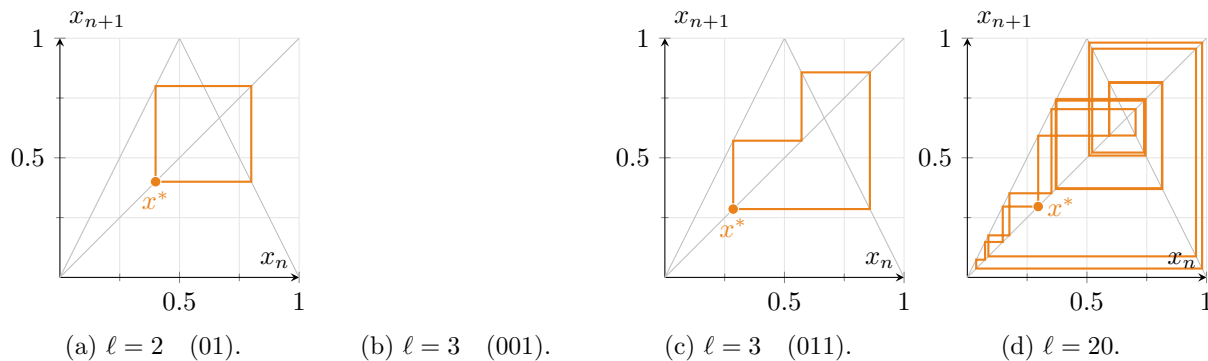


Fig. 5: Return map of UPP  $x^*$  with period- $\ell$  in tent map with  $r = 2$ .

### 4.3 Bernoulli map

Let  $F(x) = 2x \bmod 1$  be the Bernoulli map, Tab. IV shows the results of experiment. The values in the column “fraction” are the approximated fractions for  $x^*$ . We can see that they are equivalent to the analytically derived periodic points. As a result, we have found that our method is adaptive not only to convex upward maps, like logistic or tent maps, but also to general two-to-one mappings.

Table IV: Numerical result for Bernoulli map with  $x_0 = 0.7$ .

$\ell$	basic sequence	$x^*$	fraction	error	$N$	$\nu$
2	01	0.333 333 333 333	1/3	$1.0 \times 10^{-15}$	24	$2.500 \times 10^{-1}$
3	001	0.142 857 142 857	1/7	$2.2 \times 10^{-16}$	17	$1.250 \times 10^{-1}$
	011	0.428 571 428 571	3/7	$8.3 \times 10^{-16}$	16	$1.250 \times 10^{-1}$
20	01101110001110111000	0.430 595 808 597	150504/349525	0.0	3	$9.537 \times 10^{-7}$

## 5. Conclusion

In this study, we have focused on the two-to-one maps and developed the numerical method to calculate the UPPs, based on the theory of the symbolic dynamical system. The core technique is the definition of the non-deterministic map  $G$ . From the experimental results, we have confirmed the method works very well with the following advantages: the method converges rapidly as  $\ell$  is larger; we can find the UPPs that are always unstable in the given parameter range. Symbolic dynamical system and the asymptotic stability of UPP for  $G$  guarantees the convergence of the method. Hereby, the convergence error is scalable depending on the numeric precision of the software.

In future works, we would like to expand the method to the higher dimension cases.

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