A NOTE ON SPLITTING NUMBERS FOR GALOIS COVERS AND π_1-EQUIVALENT ZARISKI k-PLETS

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ABSTRACT. In this paper, we introduce *splitting numbers* of subvarieties in a smooth complex variety for a Galois cover, and prove that the splitting numbers are invariant under certain homeomorphisms. In particular cases, we show that splitting numbers enable us to distinguish the topology of complex plane curves even if the fundamental groups of the complements of plane curves are isomorphic. Consequently, we prove that there are π_1-equivalent Zariski k-plets for any integer $k \geq 2$.

INTRODUCTION

In this paper, we investigate the topology of plane curves by technique of algebraic geometry, where the *topology of a plane curve* means the homeomorphism class of the pair of the complex projective plane $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$ and a reduced divisor on $\mathbb{P}^2$. The *combinatorial data* of a plane curve consists of data such as the number of its irreducible components, the degree and the types of singularities of each irreducible component and the intersection data of the irreducible components (see [3] for details). It is known that the combinatorial data of a plane curve determine the topology of a tubular neighborhood of the plane curve (cf. [3]). In 1929 [18], O. Zariski proved that the fundamental group of the complement of a 6-cuspidal plane sextic is the free product of two cyclic groups of order 2 and 3 if the 6 cusps are on a conic, and the cyclic group of order 6 otherwise. This result [18] showed that the combinatorial data of a plane curve does not determine the topology of the plane curve. We call a k-plet $(C_1, \ldots, C_k)$ of plane curves $C_i \subset \mathbb{P}^2$ a *Zariski k-plet* if $C_1, \ldots, C_k$ have same combinatorial data, and there exist no homeomorphisms $h_{ij} : \mathbb{P}^2 \to \mathbb{P}^2$ satisfying $h_{ij}(C_j) = C_i$ for any i and j with $i \neq j$. In the case of $k = 2$, a Zariski 2-plet is called a *Zariski pair*.

From 90's, many examples of Zariski pairs were constructed (for example, see [1, 3, 5, 8, 14, 17]). The main tool to distinguish the topology of plane curves is the fundamental groups as [18]. Indeed, some topological invariants (for example, Alexander polynomials, characteristic varieties, existence/non-existence of certain Galois covers branched along given curves) are used to distinguish the fundamental groups of the complements of plane curves. Some authors introduced other methods to distinguish the topology of plane curves. For example, Artal–Carmona–Cogolludo [2] proved that the braid monodromy of an affine plane curve determine the topology of a related projective plane curve. A. Degtyarëv and I. Shimada constructed Zariski pairs of simple sextic curves by the theory of $K3$-surfaces given

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by double covers branched at simple sextic curves (for example [9, 10, 16]). Artal–
Florens–Guerville [4] introduced a new topological invariant, called $\mathcal{I}$-invariant, for
a line arrangement $\mathcal{A}$ derived from the peripheral structure $\pi_1(B_{\mathcal{A}}) \to \pi_1(E_{\mathcal{A}})$,
where $B_{\mathcal{A}}$ and $E_{\mathcal{A}}$ are the boundary manifold and the exterior of $\mathcal{A}$ respectively,
and constructed Zariski pairs of line arrangements in [4, 11]. Recently, Guerville–
Meilhan [12] generalized $\mathcal{I}$-invariant of line arrangements to an invariant of any
algebraic curves, called *linking set*. Artal–Tokunaga [5] and Shimada [16] have
studied *splitting curves* with respect to double covers to construct Zariski k-plets.
After these works [5, 16], S. Bannai [6] introduced *splitting type* with respect to
double covers, and he and the author constructed Zariski pairs and 3-plets by using
splitting type in [6, 7]. In particular, Degtyarëv [10] found a Zariski pair (C_1, C_2)
of simple sextic curves C_1 and C_2 such that the fundamental groups of the com-
plements of C_1 and C_2 are isomorphic, such a Zariski pair called a π_1-*equivalent
Zariski pair*. Moreover, in [3], a Zariski pair (C_1, C_2) such that $\mathbb{P}^2 \setminus C_1$ and $\mathbb{P}^2 \setminus C_2$
are homeomorphic, called a *complement-equivalent Zariski pair*, was constructed
by using the braid monodromy as [2]. However, it seems that there are not many
examples of π_1-equivalent Zariski pairs. On the other hand, Shimada [15] con-
structed families of equisingular curves with many connected components, which
can not be distinguished by the fundamental groups. It is not known whether two
equisingular curves in distinct connected components of a family of [15] provide a
Zariski pair, or not. In the present paper, we introduce a *splitting number* of an
irreducible subvariety in a smooth variety for a Galois cover, which is inspired by
the splitting type of Bannai [6], and prove that a family of Shimada [15] provides
a Zariski k-plet.

To state the main theorem, we recall the families of Shimada [15]. In [15],
Shimada defined plane curves of type (b, m) as follows.

Definition 0.1 ([15, Definition 1.1]). Let b and m be positive integers such that
$b \geq 3$ and $b \equiv 0 \pmod{m}$. Put $n := b/m$. A projective plane curve $R \subset \mathbb{P}^2$ is said
to be *of type* (b, m) if it satisfies the followings;

(1) R consists of two irreducible components B and E of degree b and 3, re-
spectively,
(2) both of B and E are non-singular,
(3) the set-theoretical intersection of B and E consists of $3n$ points, and
(4) at each intersection point, B and E intersect with multiplicity m.

He considered the family $\mathcal{F}_{b,m} \subset \mathbb{P}_* H^0(\mathbb{P}^2, \mathcal{O}(b+3))$ of all curves of type (b, m).
Here $\mathbb{P}_* H^0(\mathbb{P}^2, \mathcal{O}(d))$ is the projective space of one-dimensional subspaces of the
vector space $H^0(\mathbb{P}^2, \mathcal{O}(d))$, which parameterize all plane curves of degree d. He
proved the following theorem.

Theorem 0.2 ([15, Theorem 1.2]). *Suppose that $b \geq 4$, and let m be a divisor of*
b.

(1) *The number of the connected components of $\mathcal{F}_{b,m}$ is equal to the number of*
divisors of m.
(2) *Let R be a member of $\mathcal{F}_{b,m}$. Then the fundamental group $\pi_1(\mathbb{P}^2 \setminus R)$ is*
isomorphic to $\mathbb{Z}$ if b is not divisible by 3, while it is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
if b is divisible by 3.

Hence, if R_1 and R_2 are curves of type (b, m) in distinct connected components of
$\mathcal{F}_{b,m}$, the embeddings of R_1 and R_2 in $\mathbb{P}^2$ can not be distinguished topologically by

fundamental groups. Note that, if R_1 and R_2 are in a same connected component of $\mathcal{F}_{b,m}$, then (R_1, R_2) is not a Zariski pair. The following problem was raised.

Problem 0.3. For two plane curves R_1 and R_2 of type (b, m) in distinct connected components of $\mathcal{F}_{b,m}$, is the pair (R_1, R_2) a Zariski pair?

In the present paper, we introduce a *splitting number* of a subvariety in a smooth variety for a Galois cover (Definition 1.1), and prove that it is invariant under certain homeomorphisms (Proposition 1.3). Here we consider the splitting number of a subvariety for a Galois cover as an invariant of the algebraic subset consisting of the subvariety and the branch locus of the Galois cover, and a certain homeomorphism means a homeomorphism from the ambient space to itself which does not interchange the subvariety and any component of the branch locus and keeps the Galois cover (see Proposition 1.3 for details). The main theorem is the following theorem.

Theorem 0.4. *Let $b \geq 4$ be an integer, and let m be a divisor of b, and let R_1 and R_2 be plane curves of type (b, m). Then (R_1, R_2) is a Zariski pair if and only if R_1 and R_2 are in distinct connected components of $\mathcal{F}_{b,m}$.*

By Theorem 0.2 and 0.4, we obtain the following corollary.

Corollary 0.5. *For any integer $k \geq 2$, there exists a π_1-equivalent Zariski k-plet.*

The present paper is organized as follows. In the first section, we define splitting numbers of subvarieties in a smooth variety for a Galois cover, and prove that the splitting numbers are invariant under certain homeomorphisms. In the second section, we investigate the splitting numbers for simple cyclic covers, and give a method of computing the splitting numbers of smooth plane curves for simple cyclic covers. In the third section, we recall the connected components of $\mathcal{F}_{b,m}$ given in [15]. In the final section, we prove Theorem 0.4 and Corollary 0.5 by using splitting numbers.

1. Splitting numbers of subvarieties for Galois covers

Let Y be a smooth variety. Let $B \subset Y$ be a reduced divisor, and let G be a finite group. A surjective homomorphism $\theta : \pi_1(Y \setminus B) \twoheadrightarrow G$ induces an étale G-cover $\phi' : X' \to Y \setminus B$, where a *G-cover* is a Galois cover $\phi : X \to Y$ with $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$. Hence we obtain an extension of rational function fields $\mathbb{C}(X')/\mathbb{C}(Y)$. By $\mathbb{C}(X')$-normalization of Y, π' is extended to a branched G-cover $\pi : X \to Y$ uniquely. We call $\phi : X \to Y$ the G-cover induced by the surjection $\theta : \pi_1(Y \setminus B) \twoheadrightarrow G$.

Definition 1.1. Let Y be a smooth variety, and let $\phi : X \to Y$ be a G-cover branched at a reduced divisor $B \subset Y$ for a finite group G. For an irreducible subvariety $C \subset Y$ with $C \not\subset B$, we call the number of irreducible components of $\phi^* C$ the *splitting number of C for ϕ*, and denote it by $s_\phi(C)$. If $s_\phi(C) \geq 2$, we call C a *splitting subvariety* (a *splitting curve* if $\dim_{\mathbb{C}} C = 1$) for ϕ.

Remark 1.2. Let $C_1, C_2, B \subset \mathbb{P}^2$ be three plane curves of degree $d_1, d_2, 2n$, respectively, and let $\phi : S' \to \mathbb{P}^2$ be the double cover branched along B. Assume that $C_i \not\subset B$ and $s_\phi(C_i) = 2$ for $i = 1, 2$, say $\phi^* C_i = C_i^+ + C_i^-$. In [6], Bannai defined *splitting type* for the triple $(C_1, C_2; B)$ as the pair (m_1, m_2) of the intersection

numbers $C_1^+.C_2^+ = m_1$ and $C_1^+.C_2^- = m_2$ for suitable choice of labels. More-over, he proved that splitting types are invariant under certain homeomorphisms [6, Proposition 2.5].

We prove that splitting numbers are invariant under certain homeomorphisms by the same idea of the proof of [6, Proposition 2.5].

Proposition 1.3. *Let Y be a smooth variety. Let $B_1, B_2 \subset Y$ be two reduced divisors, and let G be a finite group. For $i = 1, 2$, let $\phi_i : X_i \to Y$ be the G-cover induced by a surjection $\theta_i : \pi_1(Y \setminus B) \twoheadrightarrow G$. Assume that there exists a homeomorphism $h : Y \to Y$ with $h(B_1) = B_2$ and an automorphism $\sigma : G \to G$ and $\sigma \circ \theta_2 \circ h_* = \theta_1$. Then the followings hold;*
  (1) *there exists a unique G-equivariant homeomorphism $\tilde{h} : X_1' \to X_2'$ with $\phi_2 \circ \tilde{h} = h \circ \phi_1$ up to G-action, where $X_i' = X_i \setminus \phi_i^{-1}(B_i)$ for $i = 1, 2$, and*
  (2) *for two irreducible subvarieties $C_1, C_2 \subset Y$ with $C_2 = h(C_1)$ and $C_1 \not\subset B_1$, $\tilde{h}$ induces a one to one correspondence between the set of irreducible components of $\phi_1^* C_1$ to the set of those of $\phi_2^* C_2$. Moreover, this correspondence is given by*

$$\widetilde{C}_1 \mapsto \overline{\tilde{h}\left(\widetilde{C}_1 \setminus \phi_1^{-1}(B_1)\right)}$$

  for an irreducible component $\widetilde{C}_1$ of $\phi_1^* C_1$, where $\overline{S}$ is the closure of a subset S of X_2. In particular, $s_{\phi_1}(C_1) = s_{\phi_2}(C_2)$.

Proof. The assertion (1) is clear by the uniqueness of the G-covers induced by $\theta_i : \pi_1(Y \setminus B_i) \twoheadrightarrow G$. We prove the assertion (2). Put $C_i' := C_i \setminus (B_i \cup \mathrm{Sing}(C_i))$ and let $\phi_i^* C_i = \sum_{j=1}^{n_i} \widetilde{C}_{ij}$ be the irreducible decomposition for each $i = 1, 2$, where $n_i = s_{\phi_i}(C_i)$. Since $C_i \setminus (B_i \cup \mathrm{Sing}(C_i))$ is connected and $\widetilde{C}_{ij}' := \widetilde{C}_{ij} \setminus \phi_i^{-1}(B_i \cup \mathrm{Sing}(C_i))$ is smooth, $\widetilde{C}_{ij}'$ is a connected component of $\phi_i^{-1}(C_i')$. Since $\tilde{h} : X_1' \to X_2'$ is a homeomorphism with $\phi_2 \circ \tilde{h} = h \circ \phi_1$, $\tilde{h}|_{\phi^{-1}(C_1')} : \phi^{-1}(C_1') \to \phi_2^{-1}(C_2')$ is a homeomorphism. Hence the numbers of irreducible components of $\phi_i^* C_i$ ($i = 1, 2$) coincide, $n_1 = n_2$. Moreover, since the closure of $\widetilde{C}_{ij}'$ is $\widetilde{C}_{ij}$, the assertion (2) holds. $\square$

For a smooth curve $B \subset \mathbb{P}^2$ of degree b and a divisor m of b, a surjection $\pi_1(\mathbb{P}^2 \setminus B) \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$ is uniquely determined since $\pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/b\mathbb{Z}$. Hence, by Proposition 1.3, we obtain the following corollary.

Corollary 1.4. *Let $B_1, B_2, C_1, C_2 \subset \mathbb{P}^2$ be smooth curves with $b = \deg B_1 = \deg B_2$, and let m be a divisor of b. Put $\phi_i : X_i \to \mathbb{P}^2$ the induced $\mathbb{Z}/m\mathbb{Z}$-cover by $\pi_1(\mathbb{P}^2 \setminus B_i) \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$ for each $i = 1, 2$. If there exists a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ with $h(B_1) = B_2$ and $h(C_1) = C_2$, then $s_{\phi_1}(C_1) = s_{\phi_2}(C_2)$.*

Remark 1.5. Under the assumption of Corollary 1.4, the splitting number $s_{\phi_i}(C_i)$ of C_i for ϕ_i is equal to the index of the image of the homomorphism

$$H_1(C_i \setminus B_i, \mathbb{Z}) \xrightarrow{\iota_*} H_1(\mathbb{P}^2 \setminus B_i, \mathbb{Z}) \cong \mathbb{Z}/b\mathbb{Z} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z},$$

where ι is the inclusion map $C_i \setminus B_i \hookrightarrow \mathbb{P}^2 \setminus B_i$ (see [13]). Moreover, in [13], Guervillé and the author prove equivalence between the splitting number and the linking invariant of [12] in this case. In Section 2, we compute the splitting number by algebro-geometric approach.

2. SPLITTING NUMBERS FOR SIMPLE CYCLIC COVERS

In this section, we investigate a method of computing the splitting number of a smooth plane curve for a simple cyclic cover, where a simple cyclic cover is defined in Definition 2.1 below. In this section, let Y be a smooth surface. For a line bundle $\mathcal{L}$ on Y, let $p_{\mathcal{L}} : T_{\mathcal{L}} \to Y$ denote the total space associated to $\mathcal{L}$.

Definition 2.1. Let B be either a reduced curve or the zero divisor on Y. Assume that there exists a line bundle $\mathcal{L}$ on Y such that $\mathcal{O}_Y(B) \cong \mathcal{L}^{\otimes n}$. Let $s \in H^0(Y, \mathcal{O}_Y(B))$ be a section vanishing exactly along B. A cyclic cover $\phi : X \to Y$ of degree n is a *simple cyclic cover branched along B* if X is isomorphic to the subvariety of $T_{\mathcal{L}}$ defined by $p_{\mathcal{L}}^* s - t^n = 0$ and ϕ coincides with the restriction of $p_{\mathcal{L}}$ to the subviriety, where $t \in H^0(T_{\mathcal{L}}, p_{\mathcal{L}}^* \mathcal{L})$ is the tautological section.

Definition 2.2. Let $\phi : X \to Y$ be a Galois cover branched along $B \subset Y$. Let $C \subset Y$ be an irreducible curve of Y. Let $\widetilde{C}_0$ denote an irreducible component of $\phi^* C$, and let $\bar{\eta}_0 : \overline{C}_0 \to \widetilde{C}_0$ and $\eta : \overline{C} \to C$ be the normalizations. We say that ϕ is *essentially unramified over C* if the cover $\phi_C : \overline{C}_0 \to \overline{C}$ induced by $\phi \circ \bar{\eta}_0$ is unramified, and *essentially ramified over C* otherwise.

Remark 2.3. Let G be a finite abelian group. Let $\phi : X \to Y$ be a G-cover over Y, and let C be an irreducible curve on Y. The cover $\phi_C : \overline{C}_0 \to \overline{C}$ induced by ϕ is a G_0-cover since G is abelian, where $G_0 \subset G$ is the stabilizer of $\widetilde{C}_0$. If ϕ is essentially ramified over C, then the quotient X_1 of X by the subgroup of G, which is generated by all stabilizers of ramification points of $\phi_C : \overline{C}_0 \to \overline{C}$, provides an abelian cover $\phi_1 : X_1 \to Y$ which is essentially unramified over C such that $s_\phi(C) = s_{\phi_1}(C)$. Hence, to compute the splitting number $s_\phi(C)$, we may assume that ϕ is essentially unramified over C if $\phi : X \to Y$ is an abelian cover.

Let $\phi : X \to Y$ be a simple cyclic cover of degree m branched along $B \subset Y$, and let $C \subset Y$ be an irreducible curve with $C \not\subset B$. We consider the splitting number $s_\phi(C)$. For an intersection $P \in B \cap C$ and a local branch ℓ of C at P, let $\mathrm{I}_{P,\ell}$ denote the local intersection number of B and ℓ at P. Let $\sigma : \widehat{Y} \to Y$ be a succession of blowing-ups such that $\sigma^{-1}(B + C)$ is simple normal crossing. Let $\widehat{C} \subset \widehat{Y}$ denote the strict transformation of C by σ.

Lemma 2.4. *With the above assumption, let $E_{P,\ell}$ be the irreducible component of $\sigma^* B$ which intersects with the local branch $\hat{\ell}$ of $\widehat{C}$ corresponding to ℓ. Then the multiplicity of $E_{P,\ell}$ in $\sigma^* B$ is equal to $I_{P,\ell}$.*

Proof. Since the multiplicity of $E_{P,\ell}$ in $\sigma^* B$ is depend only on the singularities of ℓ and B at P, we may assume that C is locally irreducible at P. Let $\sigma_P : \widehat{Y}_P \to Y$ be the succession of blowing-ups over P in σ, and let $\widehat{C}_P$ be the strict transform of C by σ_P. Let $E'_{P,\ell}$ be the irreducible component of $\sigma_P^* B$ which intersects with $\widehat{C}_P$, and let $m_{P,\ell}$ be the multiplicity of $E'_{P,\ell}$ in $\sigma_P^* B$. It is sufficient to prove that $m_{P,\ell} = \mathrm{I}_{P,\ell}$. Note that the exceptional set $\sigma_P^{-1}(P)$ intersects with $\widehat{C}_P$ at one point.

By the projection formula, we obtain

$$m_{P,\ell} = \sigma_P^* B.\widehat{C}_P - \sum_{(P',\ell')\neq(P,\ell)} \mathrm{I}_{P',\ell'}$$

$$= B.C - \sum_{(P',\ell')\neq(P,\ell)} \mathrm{I}_{P',\ell'}$$

$$= \mathrm{I}_{P,\ell}. \qquad\qquad \square$$

We may assume that $E.\widehat{C} \leq 1$ for each irreducible component E of $\sigma^* B$ after more blowing-ups if necessary. Let $\hat{\phi} : \widehat{X} \to \widehat{Y}$ be the $\mathbb{C}(X)$-normalization of $\widehat{Y}$. In general, $\hat{\phi}$ is not a simple cyclic cover. By Lemma 2.4, if there is a local branch ℓ of C at $P \in B \cap C$ such that m is not a divisor of $\mathrm{I}_{P,\ell}$, then $\hat{\phi}$ is branched along $E_{P,\ell}$, hence ϕ is essentially ramified over C. By Remark 2.3, we may assume

$$\mathrm{I}_{P,\ell} \equiv 0 \pmod{m}$$

for any local branch ℓ of C at any intersection point $P \in B \cap C$. Let L be a divisor on Y such that the line bundle $\mathcal{L}$ associated to L defines $\phi : X \to Y$ as in Definition 2.1, and let $D_{B,C}$ denote the following divisor on $\widehat{C}$;

$$D_{B,C} := \frac{1}{m}(\sigma^* B)|_{\widehat{C}} = \left(\frac{1}{m} \sum_{(P,\ell)} \mathrm{I}_{P,\ell} E_{P,\ell} \right)\Bigg|_{\widehat{C}},$$

where the summand runs over all intersections $P \in B \cap C$ and all local branches ℓ of C at P. Put $D'_{B,C} := (\sigma^* L)|_{\widehat{C}} - D_{B,C}$. Note that, by regarding $\widehat{C}$ as the smooth model of C, $D_{B,C}$ does not depend on choice of σ by Lemma 2.4. Moreover, $mD'_{B,C}$ is linearly equivalent to 0 on $\widehat{C}$.

Proposition 2.5. *Under the above circumstance, $s_\phi(C) = \nu$ if and only if the order of $[\mathcal{O}_{\widehat{C}}(D'_{B,C})] \in \mathrm{Pic}^0(\widehat{C})$ is equal to m/ν.*

Proof. We put

$$\widehat{\mathcal{L}} := \mathcal{O}_{\widehat{Y}}(\sigma^* L - \frac{1}{m} \sum \mathrm{I}_{P,\ell} E_{P,\ell}).$$

Let $\hat{\phi} : \widehat{X} \to \widehat{Y}$ be the $\mathbb{C}(X)$-normalization of $\widehat{Y}$, and put

$$\widehat{Y}' := \widehat{Y} \setminus \mathrm{Supp}(\sigma^* B - \sum \mathrm{I}_{P,\ell} E_{P,\ell}) \quad \text{and} \quad \widehat{X}' := \hat{\phi}^{-1}(\widehat{Y}').$$

Note that $\widehat{C} \subset \widehat{Y}'$. The restriction of $\hat{\phi}$ to $\widehat{X}'$, $\hat{\phi}' : \widehat{X}' \to \widehat{Y}'$, is an étale simple cyclic cover of degree m defined in $T_{\widehat{\mathcal{L}}}$ over $\widehat{Y}'$. By Stein factorization of $\sigma \circ \hat{\phi} : \widehat{X} \to Y$, we obtain a birational morphism $\tilde{\sigma} : \widehat{X} \to X$ with $\phi \circ \tilde{\sigma} = \sigma \circ \hat{\phi}$. The birational morphism $\tilde{\sigma}$ provides a one to one correspondence between irreducible components of $\hat{\phi}^* \widehat{C}$ and those of $\phi^* C$. Thus we have $s_\phi(C) = s_{\hat{\phi}}(\widehat{C}) = s_{\hat{\phi}'}(\widehat{C})$. Since the restriction of $T_{\widehat{\mathcal{L}}}$ over $\widehat{C}$ is $T_{\mathcal{O}(D'_{B,C})}$, the assertion follows from the next lemma. $\square$

Lemma 2.6. *Let C be a smooth variety, and let $\mathcal{L}$ be a line bundle on C with $\mathcal{L}^{\otimes\mu} \cong \mathcal{O}_C$ and $\mathcal{L}^{\otimes i} \not\cong \mathcal{O}_C$ for $1 \leq i < \mu$. Put $m := \mu\nu$ for some $\nu \in \mathbb{Z}_{>0}$. Let $\widetilde{C}$ be the closed subset of $T_{\mathcal{L}}$ defined by $t^m - 1 = 0$, where $t \in H^0(T_{\mathcal{L}}, p_{\mathcal{L}}^* \mathcal{L})$ is the tautological section. Then the number of connected components of $\widetilde{C}$ is equal to ν.*

Proof. Since $\mathcal{L}^{\otimes \mu} \cong \mathcal{O}_C$, $t^\mu - \zeta_\nu^j = 0$ defines a closed subset of $T_\mathcal{L}$ for $1 \le j < \nu$, where ζ_ν is a primitive ν-th root of unity. Since $\mathcal{L}^{\otimes i} \not\cong \mathcal{O}_C$ for $1 \le i < \mu$, $t^i - a = 0$ does not define globally a closed subset of $T_\mathcal{L}$. Thus the number of connected components of $\widetilde{C}$ is equal to ν. $\qquad\square$

In general, it seems difficult to compute the order of $[\mathcal{O}_{\widehat{C}}(D'_{B,C})] \in \mathrm{Pic}^0(\widehat{C})$. However, the following theorem provides a method of computing splitting numbers of smooth plane curves for simple cyclic covers.

Theorem 2.7. *Let $\phi : X \to \mathbb{P}^2$ be a simple cyclic cover of degree m branched along a plane curve B of degree $b = mn$, and let $C \subset \mathbb{P}^2$ be a smooth curve of degree d. Assume that $\mathrm{I}_P \equiv 0 \pmod{m}$ for each $P \in B \cap C$, where I_P is the local intersection multiplicity of B and C at P. Let ν be a divisor of m, say $m = \mu\nu$. Then $s_\phi(C) = \nu$ if and only if the following conditions hold;*
  1. *for $1 \le k < \mu$, there are no curves $D_{kn} \subset \mathbb{P}^2$ of degree kn such that $D_{kn}|_C = k D_{B,C}$, where $D_{B,C}$ is regarded as a divisor on C;*
  2. *there exists a curve $D_{\mu n} \subset \mathbb{P}^2$ of degree μn such that $D_{\mu n}|_C = \mu D_{B,C}$.*

Proof. Let $f = 0$ be a defining equation of $C \subset \mathbb{P}^2$. We have the following exact sequence;

$$0 \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(kn - d)) \xrightarrow{\times f} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(kn)) \xrightarrow{\alpha} H^0(C, \mathcal{O}_C(kn)) \to 0,$$

where α is the restriction to C. By Proposition 2.5, $s_\phi(C) = \nu$ if and only if the order of $[\mathcal{O}_C(D'_{B,C})] \in \mathrm{Pic}^0(C)$ is μ. The surjection α implies that the order of $[\mathcal{O}_C(D'_{B,C})]$ is μ if and only if the conditions (1) and (2) hold. $\qquad\square$

3. Plane curves of type (b,m)

In this section, we recall the equisingular families of plane curves given in [15]. Let b be a positive integer with $b \ge 3$, and let m be a divisor of b. We put $n := b/m$. Let $\mathcal{F}_{b,m} \subset \mathbb{P}_* H^0(\mathbb{P}^2, \mathcal{O}(b+3))$ be the family of all curves of type (b,m). Note that any two curves R and R' of type (b,m) have same combinatorics. Let $R = B + E$ be a curve of type (b,m). Let D_R denote the reduced divisor $(B|_E)_{\mathrm{red}}$ of degree $3n$ on E, and let H denote a divisor of degree 3 on E that is obtained as the intersection of E and a line on $\mathbb{P}^2$. Then $\mathcal{O}_E(D_R - nH)$ is an invertible sheaf of degree 0 on E;

$$[\mathcal{O}_E(D_R - nH)] \in \mathrm{Pic}^0(E).$$

Note that D_R and $D_R - nH$ correspond to $D_{B,E}$ and $D'_{B,E}$ in Section 2, respectively. Let $\lambda(R)$ be the order of the isomorphism class $[\mathcal{O}_E(D_R - nH)]$ in $\mathrm{Pic}^0(E)$, which is a divisor of m. For a divisor μ of m, we write by $\mathcal{F}_{b,m}(\mu)$ the union of all connected components of $\mathcal{F}_{b,m}$ on which the function λ is constantly equal to μ.

$$\mathcal{F}_{b,m} = \coprod_{\mu | m} \mathcal{F}_{b,m}(\mu)$$

Shimada [15] proved the following proposition.

Proposition 3.1 ([15, Proposition 2.1]). *Suppose that $b \ge 3$, and let m be a divisor of b. For any divisor μ of m, the variety $\mathcal{F}_{b,m}(\mu)$ is irreducible and of dimension $(b-1)(b-2)/2 + 3n + 8$.*

Remark 3.2. Proposition 3.1 implies that the number of connected components of $\mathcal{F}_{b,m}$ is equal to the number of divisors of m.

4. Proofs

In this section, we prove Theorem 0.4 and Corollary 0.5.

Proof of Theorem 0.4. Let μ_1 and μ_2 be distinct divisors of m, and let $R_i = B_i + E_i$ be a member of $\mathcal{F}_{b,m}(\mu_i)$ for each $i = 1, 2$. Let $\phi_i : X_i \to \mathbb{P}^2$ be the simple cyclic cover of degree m branched along B_i for each $i = 1, 2$. If there exists a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h(R_1) = R_2$, then $h(B_1) = B_2$ and $h(E_1) = E_2$ since $\deg B_i = b > 3 = \deg E_i$. By Corollary 1.4, we obtain $s_{\phi_1}(E_1) = s_{\phi_2}(E_2)$. On the other hand, by Proposition 2.5 and the definition of $\mathcal{F}_{b,m}(\mu_i)$, we have $s_{\phi_i}(E_i) = m/\mu_i$ for each $i = 1, 2$. Hence (R_1, R_2) is a Zariski pair.

Conversely, if $\mu_1 = \mu_2$, then it is clear that there is a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h(R_1) = R_2$ since R_1 and R_2 are members of $\mathcal{F}_{b,m}(\mu_1) = \mathcal{F}_{b,m}(\mu_2)$. $\square$

Proof of Corollary 0.5. By Theorem 0.2, the number of connected components of $\mathcal{F}_{5^{k-1}, 5^{k-1}}$ is equal to k. For each integer $0 \leq i \leq k-1$, let R_i be a member of $\mathcal{F}_{5^{k-1}, 5^{k-1}}(5^i)$. By Theorem 0.2 and 0.4, $(R_0, \ldots, R_{k-1})$ is a π_1-equivalent Zariski k-plet. $\square$

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