GALOIS COVERS OF GRAPHS AND EMBEDDED TOPOLOGY OF PLANE CURVES

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Abstract. The splitting number is effective to distinguish the embedded topology of plane curves, and it is not determined by the fundamental group of the complement of the plane curve. In this paper, we give a generalization of the splitting number, called the *splitting graph*. By using the splitting graph, we classify the embedded topology of plane curves consisting of one smooth curve and non-concurrent three lines, called *Artal arrangements*.

1. Introduction

In this paper, we study the embedded topology of plane curves in the complex projective plane $\mathbb{P}^2 := \mathbb{CP}^2$, such as in knot and link theory. Here the *embedded topology* of a plane curve $\mathcal{C} \subset \mathbb{P}^2$ is the homeomorphism class of the pair $(\mathbb{P}^2, \mathcal{C})$ of $\mathbb{P}^2$ and the reduced divisor $\mathcal{C}$ on $\mathbb{P}^2$. We introduce a new invariant, called the *splitting graph*, using a Galois cover of graphs, which describe the "splitting" of a plane curve by a Galois cover of $\mathbb{P}^2$.

The first result about the embedded topology of plane curves is given by O. Zariski [26]. He studied the existence of a certain algebraic function for a given plane curve, and pointed out that the existence can be reduced to finding the fundamental group of the complement of the given curve (the word *complement* is often omitted for short). He also proved that the fundamental group of a plane sextic curve with six cusps is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ if the six cusps lies on a conic, and the fundamental group is not isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$ otherwise (in fact, it is isomorphic to $\mathbb{Z}_6$ proved by M. Oka [14]), where $\mathbb{Z}_i$ is the cyclic group of order i. This result shows that the configuration of singularities of a plane curve can affect the fundamental group (hence the embedded topology) of the plane curves.

It is known that, if two plane curves have the same embedded topology, then they have the same combinatorics, i.e., they are equisingular

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(see [3] for details). However Zariski's result shows that the converse is false. A pair of plane curves with the same combinatorics is called *Zariski pair* if they have different embedded topology. From the 90's, E. Artal [1], M. Oka [15], H. Tokunaga [23, 24], and others have discovered Zariski pairs by studying fundamental groups of plane curves. In 21st century, π_1-equivalent Zariski pairs have been discovered (cf. [8, 18]), and the following problem has arisen naturally. (Here two plane curves are said to be π_1-*equivalent* if their fundamental groups are isomorphic.)

Problem 1.1. Give a method for distinguishing the embedded topology of π_1-equivalent plane curves.

Several methods succeed in distinguishing the embedded topology of certain π_1-equivalent plane curves. For example, there are methods using the theory of $K3$ surfaces [8], the braid monodromy [2], the splitting number [18] and the linking set [10]. The methods using the theory of $K3$ surfaces and the splitting numbers are based on techniques of algebraic geometry. On the other hand, the ones using the braid monodromy and the linking set are derived from invariants of geometrical topology. In [11], B. Guerville and the author gave a relation between the splitting number and the linking set for certain plane curves.

In this paper, we give a generalization of the splitting number, called *splitting graph*. The splitting number is based on the studies of splitting curves for double covers by Artal–Tokunaga [4], Tokunaga [25] and S. Bannai [5]. Here a plane curve $\mathcal{C}\subset\mathbb{P}^2$ is said to be *splitting* for a double cover $\phi: X\to\mathbb{P}^2$ if $\phi^*(\mathcal{C})=\mathcal{C}^+ + \mathcal{C}^-$ for some two curves $\mathcal{C}^+,\mathcal{C}^-\subset X$ with no common components and $\phi(\mathcal{C}^\pm)=\mathcal{C}$. In [4], Artal and Tokunaga implicitly used splitting curves for double covers to establish the difference of the fundamental groups of plane curves. In [25], Tokunaga defined the splitting curves with respect to double covers, and study the splitting curves as an analog of elementary number theory (see [25, Remark 0.1]). In [5], Bannai introduced the *splitting type* of certain plane curves for double covers. The splitting type gives a method for distinguishing the embedded topology of plane curves without going through the fundamental groups. In [18], the author defined the *splitting number* of irreducible curves for Galois covers, and proved that the splitting number is invariant under certain homeomorphisms from $\mathbb{P}^2$ to itself based on Bannai's idea. Moreover, by using the splitting number, he distinguished the embedded topology of the π_1-equivalent equisingular curves defined by Shimada [17]. In [19], the *connected number* of plane curves (possibly reducible) for Galois covers of $\mathbb{P}^2$ was defined, which is a modification of the splitting number.

It is known that the splitting number and the connected number are not determined by the fundamental group (see [18, 19]). These results show that studying the "splitting" of plane curves for Galois covers is effective to distinguish the embedded topology. In this paper, we define the splitting graph of plane curves for Galois covers to describe more precisely how a plane curve splits by a Galois cover, which is not determined by the fundamental group (see Remark 3.5 (ii)).

As an application of the splitting graph, we classify the embedded topology of Artal arrangements, where an *Artal arrangement* is a plane curve consisting of one smooth curve and non-concurrent three lines. The name "Artal" comes from E. Artal who gave the first Zariski pair of Artal arrangements in [1]. Note that Artal arrangements have been defined in [6] and [19], and our definition is a generalization of the ones in [6] and [19]. An Artal arrangement in [6] or [19] is a plane curve consisting of one smooth curve and its non-concurrent three tangents such that each of the three tangents intersects at just one point with the smooth curve. In this paper, we define Artal arrangements of type $\mathfrak{P}$ for a triple $\mathfrak{P}$ of partitions of an integer $d \geq 3$. In our definition, the three tangents may intersect at two or more points with the smooth curve (see Section 4 for details). Moreover, we define the splitting graph associated to an Artal arrangement. By Theorem 4.3, we obtain the following theorem.

Theorem 1.2. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two Artal arrangements of type $\mathfrak{P}$ for a triple $\mathfrak{P}$ of partitions of an integer $d \geq 3$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same embedded topology if and only if the splitting graph associated to $\mathcal{A}_1$ is equivalent to the splitting graph associated to $\mathcal{A}_2$.*

This paper is organized as follows. In Section 2, we investigate branched Galois covers of graphs in order to define and distinguish the splitting graph. In particular, we define the *net voltage class* of a closed walk on a graph for a Galois cover over the graph (Definition 2.6), and give a method of distinguishing two Galois covers of a graph (Corollary 2.9). In Section 3, we define the *splitting graph* of a plane curve for a Galois cover over $\mathbb{P}^2$ as a Galois cover of certain graphs (Definition 3.1). Moreover, we introduce a method of computing net voltage classes for a cyclic cover (Theorem 3.13). In the final section, we define *Artal arrangements* of type $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ for three partitions $\mathfrak{p}_i$ of an integer $d \geq 3$ (Definition 4.1), and classify the embedded topology of Artal arrangements by using the splitting graphs (Theorem 4.3).

2. Galois covers of graphs

In this section, we consider 'branched covers' of graphs and their equivalence. Unramified covers of graphs has been investigated by Gross–Tucker [9], Stark–Terras [20, 21, 22] and so on. In this section, we investigate how to distinguish branched Galois covers over a graph.

In this paper, we assume that any *graph* is finite, i.e., the sets of vertices and edges are finite. Note that we allow a graph to be disconnected. The sets of vertices and edges of a graph $\mathcal{G}$ are denoted by $V_{\mathcal{G}}$ and $E_{\mathcal{G}}$, respectively. We consider each edge $e \in E_{\mathcal{G}}$ to have two directions, arbitrarily distinguished as the *plus direction* e^+ and the *minus derection* e^-, i.e., $e^\pm$ are two onto functions $\{0,1\} \to V_{\mathcal{G}}(e)$ such that $e^-(0) = e^+(1)$ and $e^-(1) = e^+(0)$, where $V_{\mathcal{G}}(e)$ is the endpoint set of the edge e. We call $e^\pm(0)$ and $e^\pm(1)$ the *initial vertex* and the *terminal vertex* of $e^\pm$, respectively. For two graphs $\mathcal{G}_i$ $(i = 1, 2)$, a *map* θ from $\mathcal{G}_1$ to $\mathcal{G}_2$ is a pair $\theta = (\theta_V, \theta_E)$ of maps $\theta_V : V_{\mathcal{G}_1} \to V_{\mathcal{G}_2}$ and $\theta_E : E_{\mathcal{G}_1} \to E_{\mathcal{G}_2}$ satisfying $V_{\mathcal{G}_2}(\theta_E(e)) = \theta_V(V_{\mathcal{G}_1}(e)) \subset V_{\mathcal{G}_2}$ for any $e \in E_{\mathcal{G}_1}$. A map $\theta : \mathcal{G}_1 \to \mathcal{G}_2$ of graphs is called an *isomorphism* if θ_V and θ_E are bijective.

Definition 2.1. Let $\mathcal{G}$ be a graph, and let G be a finite group. A *Galois cover* of $\mathcal{G}$ with the Galois group G (called *G-cover* for short) is a map $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$ of graphs satisfying the following conditions;

(i) G acts on $\widetilde{\mathcal{G}}$;
(ii) the quotient map $q : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}/G$ corresponds to $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$, i.e., there exists an isomorphism $i : \widetilde{\mathcal{G}}/G \to \mathcal{G}$ such that $\phi = i \circ q$;
(iii) either $E_{\mathcal{G}} = \emptyset$ or G acts on $E_{\widetilde{\mathcal{G}}}$ freely, i.e., for any $\tilde{e} \in E_{\widetilde{\mathcal{G}}}$, $g{\cdot}\tilde{e} = \tilde{e}$ if and only if $g = \mathrm{id}_G$.

A G-cover $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$ is said to be *unbranched* or *unramified* if the cardinality of $\phi_V^{-1}(v)$ is equal to the order $|G|$ of G for each $v \in V_{\mathcal{G}}$, and *branched* or *ramified* otherwise.

Remark 2.2. Let $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a G-cover.

(i) If ϕ is unramified, then it is called a *regular covering* in [9].
(ii) By Definition 2.1 (i), we have $V_{\widetilde{\mathcal{G}}}(g{\cdot}\tilde{e}) = g{\cdot}V_{\widetilde{\mathcal{G}}}(\tilde{e}) \subset V_{\widetilde{\mathcal{G}}}$.
(iii) For each $v \in V_{\mathcal{G}}$ and $e \in E_{\mathcal{G}}$, G acts transitively on the fibers $\phi_V^{-1}(v)$ and $\phi_E^{-1}(e)$, respectively, by Definition 2.1 (ii).
(iv) We assume that ϕ preserves the directions of edges, i.e., $\phi_V(\tilde{e}^+(0)) = e^+(0)$ and $\phi_V(\tilde{e}^+(1)) = e^+(1)$ for each $e \in E_{\mathcal{G}}$ and $\tilde{e} \in \phi_E^{-1}(e)$. Hence the action of G on $\widetilde{\mathcal{G}}$ also preserves the direction of edges.
(v) The action of G on $\widetilde{\mathcal{G}}$ is faithful if $E_{\mathcal{G}} \neq \emptyset$.

We define the equivalence of Galois covers of graphs with a fixed Galois group G.

Definition 2.3. Let $\mathcal{G}_i$ $(i = 1, 2)$ be two graphs, and let G be a finite group with an automorphism $\tau : G \to G$. Assume that there is an isomorphism $\theta : \mathcal{G}_1 \to \mathcal{G}_2$. Let $\phi_i = (\phi_{V,i}, \phi_{E,i}) : \widetilde{\mathcal{G}}_i \to \mathcal{G}_i$ $(i = 1, 2)$ be two G-covers. We say that $\widetilde{\mathcal{G}}_1$ and $\widetilde{\mathcal{G}}_2$ are (θ,τ)-*equivalent*, written by $\widetilde{\mathcal{G}}_1 \sim_{(\theta,\tau)} \widetilde{\mathcal{G}}_2$, if there is an isomorphism $\tilde{\theta} = (\tilde{\theta}_V, \tilde{\theta}_E) : \widetilde{\mathcal{G}}_1 \to \widetilde{\mathcal{G}}_2$ satisfying the following conditions;

(i) $\phi_2 \circ \tilde{\theta} = \theta \circ \phi_1$, i.e., $\phi_{V,2} \circ \tilde{\theta}_V = \theta_V \circ \phi_{V,1}$ and $\phi_{E,2} \circ \tilde{\theta}_E = \theta_E \circ \phi_{E,1}$;
(ii) for any $\tilde{v} \in V_{\widetilde{\mathcal{G}}_1}$ and $g \in G$, $\tilde{\theta}_V(g{\cdot}\tilde{v}) = \tau(g){\cdot}\tilde{\theta}_V(\tilde{v})$; and
(iii) for any $\tilde{e} \in E_{\mathcal{G}_1}$ and $g \in G$, $\tilde{\theta}_E(g{\cdot}\tilde{e}) = \tau(g){\cdot}\tilde{\theta}_E(\tilde{e})$.

Remark 2.4. Let $\phi_i : \widetilde{\mathcal{G}}_i \to \mathcal{G}_i$, $\tau : G \to G$ and $\theta : \mathcal{G}_1 \to \mathcal{G}_2$ be as in Definition 2.3. For $\tilde{v}_i \in V_{\widetilde{\mathcal{G}}_i}$, we put $\mathrm{Stab}_G(\tilde{v}_i) := \{g \in G \mid g{\cdot}\tilde{v}_i = \tilde{v}_i\}$. Assume that $E_{\mathcal{G}_i} = \emptyset$ for each $i = 1, 2$. Then $\widetilde{\mathcal{G}}_1 \sim_{(\theta,\tau)} \widetilde{\mathcal{G}}_2$ if and only if, for each $v_1 \in V_{\mathcal{G}_1}$ and $v_2 := \theta_V(v_1)$, $\tau(\mathrm{Stab}_G(\tilde{v}_1))$ is conjugate to $\mathrm{Stab}_G(\tilde{v}_2)$ for any $\tilde{v}_1 \in \phi_{V,1}^{-1}(v_1)$ and $\tilde{v}_2 \in \phi_{V,2}^{-1}(v_2)$. Indeed, since $E_{\widetilde{\mathcal{G}}_i} = \emptyset$, $\widetilde{\mathcal{G}}_1 \sim_{(\theta,\tau)} \widetilde{\mathcal{G}}_2$ if and only if $\phi_{V,1}^{-1}(v_1) \sim_{(\theta,\tau)} \phi_{V,2}^{-1}(v_2)$ for each $v_1 \in V_{\mathcal{G}_1}$, which is equivalent to the latter condition by regarding $\phi_{V,i}^{-1}(v_i)$ as G-sets.

In order to give a criterion for equivalence of Galois covers of graphs with edges, we introduce an invariant of Galois covers by using closed walks. A *walk* γ on a graph $\mathcal{G}$ is an alternating sequence of vertices and directed edges

$$\gamma = (v_0, e_1^{\sigma_1}, v_1, e_2^{\sigma_2}, \dots, v_{n-1}, e_n^{\sigma_n}, v_n) \quad (\sigma_i = + \text{ or } -)$$

such that $e_i^{\sigma_i}(0) = v_{i-1}$ and $e_i^{\sigma_i}(1) = v_i$, i.e., the initial and terminal vertices of $e_i^{\sigma_i}$ are v_{i-1} and v_i, respectively, for each $i = 1, \dots, n$. For a walk $\gamma = (v_0, e_1^{\sigma_1}, \dots, v_n)$, we call v_0 and v_n the *initial vertex* and the *terminal vertex* of γ, respectively, and we call the number n of edges in γ the *length* of γ. A walk γ is said to be *closed* if the initial and terminal vertices coincide. For a Galois cover $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$ and a walk $\gamma = (v_0, e_1^{\sigma_1} \dots, v_n)$ on $\mathcal{G}$, a *lift* of γ under ϕ is a walk $\tilde{\gamma} = (\tilde{v}_0, \tilde{e}_1^{\sigma_1}, \dots, \tilde{v}_n)$ on $\widetilde{\mathcal{G}}$ satisfying $\phi_V(\tilde{v}_i) = v_i$ and $\phi_E(\tilde{e}_i) = e_i$ for any $i = 0, \dots, n$.

Let $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a G-cover of a graph $\mathcal{G}$, and let $\gamma = (v_0, e_1^{\sigma_1}, \dots, v_n)$ be a closed walk on $\mathcal{G}$. We fix a vertex $\tilde{v}_0$ such that $\phi_V(\tilde{v}_0) = v_0$. We define a subset $\widetilde{V}^{\gamma}_{\phi,i}(\tilde{v}_0)$ of $\phi_V^{-1}(v_i)$ for each $i = 0, \dots, n$ as follows:

$$\widetilde{V}^{\gamma}_{\phi,0}(\tilde{v}_0) := \{\tilde{v}_0\},$$
$$\widetilde{V}^{\gamma}_{\phi,i}(\tilde{v}_0) := \left\{ \tilde{v}_i \in \phi_V^{-1}(v_i) \;\middle|\; \begin{array}{l} \tilde{e}_i^{\sigma_i}(0) \in \widetilde{V}^{\gamma}_{\phi,i-1}(\tilde{v}_0) \text{ and } \tilde{e}_i^{\sigma_i}(1) = \tilde{v}_i \\ \text{for some } \tilde{e}_i \in \phi_E^{-1}(e_i) \end{array} \right\}.$$

In other words, $\widetilde{V}^{\gamma}_{\phi,i}(\tilde{v}_0)$ is the set of vertices $\tilde{v}_i \in \phi_V^{-1}(v_i)$ such that there is a lift of the walk $(v_0, e_1^{\sigma_1}, \dots, v_i)$ under ϕ whose initial and terminal vertices are $\tilde{v}_0$ and $\tilde{v}_i$, respectively. Let $\mathrm{NV}_\phi(\gamma, \tilde{v}_0)$ be the following subset of G:

$$\mathrm{NV}_\phi(\gamma, \tilde{v}_0) := \{g \in G \mid g\cdot\tilde{v}_0 \in \widetilde{V}^{\gamma}_{\phi,n}(\tilde{v}_0)\}.$$

Note that $\widetilde{V}^{\gamma}_{\phi,n}(\tilde{v}_0) = \{g\cdot\tilde{v}_0 \mid g \in \mathrm{NV}_\phi(\gamma, \tilde{v}_0)\}$ since the action of G on $\phi_V^{-1}(v_0)$ is transitive by Condition (ii) in Definition 2.1.

Lemma 2.5. *Let $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$, G and γ be as above. For any $\tilde{v}_0 \in \phi_V^{-1}(v_0)$ and $g \in G$,*

$$\mathrm{NV}_\phi(\gamma, g\cdot\tilde{v}_0) = g\mathrm{NV}_\phi(\gamma, \tilde{v}_0)g^{-1}.$$

Proof. For any $g_0 \in G$, g_0 is an element of $\mathrm{NV}_\phi(\gamma, \tilde{v}_0)$ if and only if there exists a lift $\tilde{\gamma} = (\tilde{v}_0, \tilde{e}_1^{\sigma_1}, \dots, \tilde{v}_n)$ of γ such that $\tilde{v}_n = g_0\cdot\tilde{v}_0$. By Remark 2.2 (iii), the latter condition is equivalent to the existence of a lift $\tilde{\gamma}' = (\tilde{v}'_0, (\tilde{e}'_1)^{\sigma_1}, \dots, \tilde{v}'_n)$ of γ such that $\tilde{v}'_0 = g\cdot\tilde{v}_0$ and $\tilde{v}'_n = g\cdot\tilde{v}_n = gg_0g^{-1}\cdot\tilde{v}'_0$. Thus, $g_0 \in \mathrm{NV}_\phi(\gamma, \tilde{v}_0)$ is equivalent to $gg_0g^{-1} \in \mathrm{NV}_\phi(\gamma, g\cdot\tilde{v}_0)$. Therefore, we obtain $\mathrm{NV}_\phi(\gamma, g\cdot\tilde{v}_0) = g\mathrm{NV}_\phi(\gamma, \tilde{v}_0)g^{-1}$. □

For a subset S of a group G, we call the family of sets

$$\{gSg^{-1} \subset G \mid g \in G\} \subset 2^G$$

the *conjugacy class* of S in G. By Lemma 2.5 and Remark 2.2 (ii), the conjugacy class of the set $\mathrm{NV}_\phi(\gamma, \tilde{v}_0)$ in G does not depend on the choice of $\tilde{v}_0 \in \phi_V^{-1}(v_0)$.

Definition 2.6. Let $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$, G and γ be as above. Let $\mathrm{NV}_\phi(\gamma)$ denote the conjugacy class of $\mathrm{NV}_\phi(\gamma, \tilde{v}_0)$, i.e.,

$$\mathrm{NV}_\phi(\gamma) := \{g\mathrm{NV}_\phi(\gamma, \tilde{v}_0)g^{-1} \mid g \in G\},$$

and we call $\mathrm{NV}_\phi(\gamma)$ the *net voltage class* of γ for ϕ.

Remark 2.7. The name "net voltage class" is derived from the net voltage on a walk in the base space of a voltage graph. Voltage graphs correspond to unramified covers of a graph, and the net voltage represents the terminal vertex of a lift of the walk (see [9]). In the case of unramified covers, a lift of a walk is uniquely determined by its initial vertex. However a lift of a walk is not uniquely determined by its initial vertex in the case of branched covers. Hence a net voltage is not uniquely determined.

For a walk $\gamma = (v_0, e_1^{\sigma_1}, \dots, v_n)$ on a graph $\mathcal{G}_1$ and a map $\theta : \mathcal{G}_1 \to \mathcal{G}_2$, let $\theta(\gamma)$ denote the following walk on $\mathcal{G}_2$:

$$\theta(\gamma) := (\theta_V(v_0), \theta_E(e_1)^{\sigma'_1}, \dots, \theta_V(v_n)),$$

where σ'_i is the sign $\pm$ such that $\theta_E(e_i)^{\sigma'_i}(0) = \theta_V(v_{i-1})$ and $\theta_E(e_i)^{\sigma'_i}(1) = \theta_V(v_i)$. The net voltage class is invariant under equivalence of Galois covers of graphs by the following proposition.

Proposition 2.8. *Let G be a finite group, let $\phi_i : \widetilde{\mathcal{G}}_i \to \mathcal{G}_i$ be a G-cover of graphs for each $i = 1, 2$, and let γ_i be a closed walk on $\mathcal{G}_i$. Assume that $\widetilde{\mathcal{G}}_1 \sim_{(\theta,\tau)} \widetilde{\mathcal{G}}_2$, where $\theta : \mathcal{G}_1 \to \mathcal{G}_2$ and $\tau : G \to G$ are an isomorphism of graphs and an automorphism of G, respectively. If $\theta(\gamma_1) = \gamma_2$, then $\tau(\mathrm{NV}_{\phi_1}(\gamma_1)) = \mathrm{NV}_{\phi_2}(\gamma_2)$.*

Proof. Let $\tilde{\theta} : \widetilde{\mathcal{G}}_1 \to \widetilde{\mathcal{G}}_2$ be an isomorphism satisfying Conditions (i), (ii) and (iii) in Definition 2.3. By Condition (i) in Definition 2.3, $\tilde{\theta}_V$ gives a bijection from $\phi_{V,1}^{-1}(v)$ to $\phi_{V,2}^{-1}(\theta_V(v))$ for each $v \in V_{\mathcal{G}_1}$. Let v_0 and w_0 be the initial vertices of γ_1 and γ_2, respectively, and let n be the length of γ_1 and γ_2. Let $\tilde{v}_0$ be a vertex in $\phi_1^{-1}(v_0)$, and put $\tilde{w}_0 := \tilde{\theta}_V(\tilde{v}_0)$. Since $\tilde{\theta}$ is an isomorphism, an sequence $(\tilde{v}_0, \tilde{e}_1^{\sigma_1}, \dots, \tilde{v}_n)$ with $\tilde{v}_i \in V_{\widetilde{\mathcal{G}}_1}$ and $\tilde{e}_i \in E_{\widetilde{\mathcal{G}}_1}$ is a walk on $\widetilde{\mathcal{G}}_1$ if and only if $(\tilde{w}_0, (\tilde{e}'_1)^{\sigma'_1}, \dots, \tilde{w}_n)$ is a walk on $\widetilde{\mathcal{G}}_2$ for certain signs σ'_i, where $\tilde{w}_i = \tilde{\theta}_V(\tilde{v}_i)$ and $\tilde{e}'_i := \tilde{\theta}_E(\tilde{e}_i)$. Hence we have $\tilde{\theta}_V(\widetilde{V}^{\gamma_1}_{\phi_1,n}(\tilde{v}_0)) = \widetilde{V}^{\gamma_2}_{\phi_2,n}(\tilde{w}_0)$. Moreover, we obtain $\tau(\mathrm{NV}_{\phi_1}(\gamma_1, \tilde{v}_0)) = \mathrm{NV}_{\phi_2}(\gamma_2, \tilde{w}_0)$ by Definition 2.3 (ii). Therefore, $\tau(\mathrm{NV}_{\phi_1}(\gamma_1)) = \mathrm{NV}_{\phi_2}(\gamma_2)$ since τ is an automorphism of G. □

A closed walk $\gamma = (v_0, e_1^{\sigma_1}, \dots, v_n)$ is said to be *simple* if $v_i \neq v_j$ for any $0 \leq i < j < n$. By the same idea of [6], we obtain the following corollary.

Corollary 2.9. *Let G be a finite group, let $\phi_i : \widetilde{\mathcal{G}}_i \to \mathcal{G}_i$ ($i = 1, 2$) be two G-covers of graphs, and let $\tau : G \to G$ be an automorphism. Let $\mathcal{W}_{\mathcal{G}_i}$ be the set of simple closed walks. If there exists no bijection $\Theta_{\mathcal{W}} : \mathcal{W}_{\mathcal{G}_1} \to \mathcal{W}_{\mathcal{G}_2}$ such that $\tau(\mathrm{NV}_{\phi_1}(\gamma_1)) = \mathrm{NV}_{\phi_2}(\Theta_{\mathcal{W}}(\gamma_1))$ for any $\gamma_1 \in \mathcal{W}_{\mathcal{G}_1}$, then there exist no isomorphisms $\theta : \mathcal{G}_1 \to \mathcal{G}_2$ such that $\widetilde{\mathcal{G}}_1 \sim_{(\theta,\tau)} \widetilde{\mathcal{G}}_2$.*

Proof. If $\theta : \mathcal{G}_1 \to \mathcal{G}_2$ is an isomorphism such that $\widetilde{\mathcal{G}}_1 \sim_{(\theta,\tau)} \widetilde{\mathcal{G}}_2$, then θ induces a bijection $\Theta_{\mathcal{W}} : \mathcal{W}_{\mathcal{G}_1} \to \mathcal{W}_{\mathcal{G}_2}$ such that $\tau(\mathrm{NV}_{\phi_1}(\gamma_1)) = \mathrm{NV}_{\phi_2}(\Theta_{\mathcal{W}}(\gamma_2))$ by Proposition 2.8. □

We investigate properties of the net voltage classes. For a walk $\gamma = (v_0, e_1^{\sigma_1}, \dots, v_n)$ with $v_n = v_0$, we call the closed walk

$$\gamma^{-1} := (v_n, e_n^{-\sigma_n}, \dots, e_1^{-\sigma_1}, v_0)$$

the *inverse walk* of γ, where $-\sigma_i := \mp$ if $\sigma_i = \pm$, respectively.

Lemma 2.10. *Let $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a G-cover of a graph $\mathcal{G}$. Let γ be a closed walk $(v_0, e_1^{\sigma_1}, \dots, v_n)$ on $\mathcal{G}$, and let $\tilde{v}_0$ be a vertex in $\phi_V^{-1}(v_0)$. Then*

$$\mathrm{NV}_\phi(\gamma^{-1}, \tilde{v}_0) = \{g^{-1} \mid g \in \mathrm{NV}_\phi(\gamma, \tilde{v}_0)\}.$$

Proof. Suppose that $g \in \mathrm{NV}_\phi(\gamma, \tilde{v}_0)$. There is a lift $\tilde{\gamma} := (\tilde{v}_0, \tilde{e}_1^{\sigma_1}, \dots, \tilde{v}_n)$ of γ such that $\tilde{v}_n = g\cdot\tilde{v}_0$. Since

$$(g^{-1}\cdot\tilde{\gamma})^{-1} = (g^{-1}\cdot\tilde{v}_n, (g^{-1}\cdot\tilde{e}_n)^{-\sigma_n}, \dots, (g^{-1}\cdot\tilde{e}_1)^{-\sigma_1}, g^{-1}\cdot\tilde{v}_0)$$

is a walk on $\widetilde{\mathcal{G}}$, we have $g^{-1} \in \mathrm{NV}_\phi(\gamma^{-1}, \tilde{v}_0)$. By the same argument, if $g \in \mathrm{NV}_\phi(\gamma^{-1}, \tilde{v}_0)$, then we obtain $g^{-1} \in \mathrm{NV}_\phi(\gamma, \tilde{v}_0)$. Therefore, the assertion holds. □

For two walks $\gamma_1 = (v_0, \dots, v_n)$ and $\gamma_2 = (w_0, \dots, w_m)$ on a graph $\mathcal{G}$ with $v_n = w_0$, let $\gamma_1\gamma_2$ denote the following walk of length $n + m$:

$$\gamma_1\gamma_2 := (v_0, \dots, v_n, w_1, \dots, w_m).$$

Lemma 2.11. *Let G be a finite group, and let $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a G-cover of a graph $\mathcal{G}$. Let γ_1 and γ_2 be two closed walks on $\mathcal{G}$ with the same initial and terminal vertex v_0, and let $\tilde{v}_0$ be a vertex in $\phi_V^{-1}(v_0)$. Then*

$$\mathrm{NV}_\phi(\gamma_1\gamma_2, \tilde{v}_0) = \mathrm{NV}_\phi(\gamma_1, \tilde{v}_0)\mathrm{NV}_\phi(\gamma_2, \tilde{v}_0),$$

where $S_1S_2 = \{g_1g_2 \mid g_1 \in S_1, g_2 \in S_2\}$ for two subsets $S_1, S_2 \subset G$.

Proof. Let g be an element of $\mathrm{NV}_\phi(\gamma_1\gamma_2, \tilde{v}_0)$. Then we have $g\cdot\tilde{v}_0 \in \widetilde{V}^{\gamma_1\gamma_2}_{\phi,m+n}(\tilde{v}_0)$. Hence there exists a lift $\tilde{\gamma} = (\tilde{v}_0, \dots, \tilde{v}_{m+n})$ of $\gamma_1\gamma_2$ with $\tilde{v}_{m+n} = g\cdot\tilde{v}_0$. Since $(\tilde{v}_0, \dots, \tilde{v}_n)$ is a lift of γ_1, there is an element $g_1 \in \mathrm{NV}_\phi(\gamma_1, \tilde{v}_0)$ such that $\tilde{v}_n = g_1\cdot\tilde{v}_0$. Since $\mathrm{NV}_\phi(\gamma_2, g_1\cdot\tilde{v}_0) = g_1\mathrm{NV}_\phi(\gamma_2, \tilde{v}_0)g_1^{-1}$ by Lemma 2.5 and $(\tilde{v}_n, \dots, \tilde{v}_{m+n})$ is a lift of γ_2, there exists an element $g_2 \in \mathrm{NV}_\phi(\gamma_2, \tilde{v}_0)$ such that

$$g\cdot\tilde{v}_0 = \tilde{v}_{m+n} = (g_1g_2g_1^{-1})\cdot\tilde{v}_n = g_1g_2\cdot\tilde{v}_0.$$

Since G acts on $\phi_V^{-1}(v_0)$ transitively, $\phi_V^{-1}(v_0)$ is isomorphic to the set of cosets G/H_1 as G-sets, where $H_1 = \{g' \in G \mid g'\cdot\tilde{v}_0 = \tilde{v}_0\}$. Hence there is an element $g' \in H_1$ such that $g = g_1 g_2 g'$. Therefore, $g \in \mathrm{NV}_\phi(\gamma_1, \tilde{v}_0) \cdot \mathrm{NV}_\phi(\gamma_2, \tilde{v}_0)$ since $g_2 g' \in \mathrm{NV}_\phi(\gamma_2, \tilde{v}_0)$.

Conversely, suppose that $g_i \in \mathrm{NV}_\phi(\gamma_i, \tilde{v}_0)$ for $i = 1, 2$. Then there are two lifts $\tilde{\gamma}_1 = (\tilde{v}_0, \ldots, \tilde{v}_n)$ and $\tilde{\gamma}_2 = (\tilde{w}_0, \ldots, \tilde{w}_m)$ of γ_1 and γ_2 such that $\tilde{w}_0 = \tilde{v}_0$, $\tilde{v}_n = g_1 \cdot \tilde{v}_0$ and $\tilde{w}_m = g_2 \cdot \tilde{w}_0$. Then the ordered set

$$\tilde{\gamma}_1(g_1 \cdot \tilde{\gamma}_2) = (\tilde{v}_0, \ldots, \tilde{v}_n, g_1 \cdot \tilde{w}_1, \ldots, g_1 \cdot \tilde{w}_m)$$

is a lift of $\gamma_1 \gamma_2$. Since $g_1 \cdot \tilde{w}_m = g_1 g_2 \cdot \tilde{v}_0$, we obtain $g_1 g_2 \in \mathrm{NV}_\phi(\gamma_1 \gamma_2, \tilde{v}_0)$. $\square$

For a closed walk $\gamma = (v_0, e_1^{\sigma_1}, \ldots, v_n)$ on a graph $\mathcal{G}$ (hence $v_0 = v_n$) and $j \in \mathbb{Z}_{\geq 0}$, let $\gamma^{(j)}$ be the following closed walk

$$\gamma^{(j)} := \left(v_{\bar{j}}, e_{\overline{j+1}}^{\sigma_{\overline{j+1}}}, \ldots, e_{\overline{j+n}}^{\sigma_{\overline{j+n}}}, v_{\overline{j+n}} \right),$$

where $\overline{j+i}$ is the integer $0 < \overline{j+i} \leq n$ with $\overline{j+i} \equiv j + i \pmod{n}$. In the case where the Galois group is abelian, the net voltage class of a closed walk does not depend on its initial vertex by the following lemma.

Lemma 2.12. *Let G be a finite abelian group, let $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a G-cover of graphs, and let γ be a closed walk on $\mathcal{G}$. Then, for any $j \in \mathbb{Z}_{\geq 0}$,*

$$\mathrm{NV}_\phi(\gamma) = \mathrm{NV}_\phi(\gamma^{(j)}).$$

Proof. Let v_0 be the initial vertex of γ. Since G is abelian, $\mathrm{NV}_\phi(\gamma, \tilde{v}_0)$ does not depend on the choice of $\tilde{v}_0 \in \phi_V^{-1}(v_0)$, and we can regard $\mathrm{NV}_\phi(\gamma)$ as a subset of G. Fix $\tilde{v}_0 \in \phi_V^{-1}(v_0)$, and take an element $g \in \mathrm{NV}_\phi(\gamma)$. Let $\tilde{\gamma} = (\tilde{v}_0, \tilde{e}_1^{\sigma_1}, \ldots, \tilde{v}_n)$ be a lift of γ under ϕ such that $\tilde{v}_n = g \cdot \tilde{v}_0$. By Remark 2.2 (ii), the ordered set

$$\tilde{\gamma}' := (\tilde{v}_{\bar{j}}, \tilde{e}_{\bar{j}+1}^{\sigma_{\bar{j}+1}}, \ldots, \tilde{v}_n, (g \cdot \tilde{e}_1)^{\sigma_1}, g \cdot \tilde{v}_1, \ldots, g \cdot \tilde{v}_{\bar{j}})$$

is a walk on $\widetilde{\mathcal{G}}$, hence it is a lift of $\gamma^{(j)}$. Thus $g \in \mathrm{NV}_\phi(\gamma^{(j)})$, and we obtain $\mathrm{NV}_\phi(\gamma) \subset \mathrm{NV}_\phi(\gamma^{(j)})$. Since $\gamma = (\gamma^{(j)})^{(n-\bar{j})}$, we also have $\mathrm{NV}_\phi(\gamma^{(j)}) \subset \mathrm{NV}_\phi(\gamma)$. $\square$

In the case of abelian covers, we may regard a closed walk γ as a class $[\gamma] := \{\gamma^{(j)} \mid j \in \mathbb{Z}_{\geq 0}\}$ in order to compute the net voltage class by Lemma 2.12. The next example shows that Lemma 2.12 fails in the case where G is not abelian.

Example 2.13. Let $\mathfrak{S}_3$ be the symmetric group of three letters, and let $\mathcal{G}$ be the complete graph of order three. We construct a branched $\mathfrak{S}_3$-cover of $\mathcal{G}$. Put $V_{\mathcal{G}} := \{a, b, c\}$ and $E_{\mathcal{G}} := \{e_{ab}, e_{bc}, e_{ca}\}$, where $e_{xy}^+(0) = x$ and $e_{xy}^+(1) = y$. Note that, since $\mathfrak{S}_3$ acts transitively on

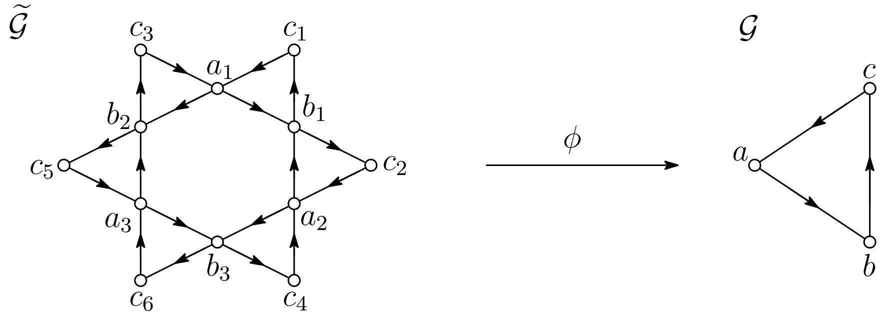

FIGURE 1. An $\mathfrak{S}_3$-cover $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$

$\phi_V^{-1}(v)$ for any $\mathfrak{S}_3$-cover $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$ and each $v \in V_{\mathcal{G}}$, we can regard $\phi_V^{-1}(v)$ as the set of cosets of a certain subgroup of $\mathfrak{S}_3$. We define a branched $\mathfrak{S}_3$-cover $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$ of $\mathcal{G}$ as follows:

Let H_1 and H_2 be the cyclic subgroups of $\mathfrak{S}_3$ generated by $(1\,2)$ and $(1\,3)$, respectively. Let a_i and b_i $(i = 1, 2, 3)$ be the cosets of $\mathfrak{S}_3/H_1$ and $\mathfrak{S}_3/H_2$, respectively, as follows:

$$\begin{aligned} a_1 &= H_1, & a_2 &= (1\,3)H_1, & a_3 &= (2\,3)H_1, \\ b_1 &= H_2, & b_2 &= (1\,2)H_2, & b_3 &= (2\,3)H_2. \end{aligned}$$

We put c_i $(i = 1, \dots, 6)$ as the elements of $\mathfrak{S}_3$ as follows:

$$c_1 = \mathrm{id}, \quad c_2 = (1\,3), \quad c_3 = (1\,2), \quad c_4 = (1\,2\,3), \quad c_5 = (1\,3\,2) \quad c_6 = (2\,3).$$

Put $V_{\widetilde{\mathcal{G}}} := \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, \dots, c_6\}$, and define $\phi_V : V_{\widetilde{\mathcal{G}}} \to V_{\mathcal{G}}$ by $\phi_V(x_i) = x$ for $x = a, b, c$. Note that $\mathfrak{S}_3$ acts on $\phi_V^{-1}(v)$ from left for each $v \in V_{\mathcal{G}}$. Assume that $e_{a_1b_1}, e_{b_1c_1}, e_{c_1a_1} \in E_{\widetilde{\mathcal{G}}}$. Then the set $E_{\widetilde{\mathcal{G}}}$ of an $\mathfrak{S}_3$-cover $\widetilde{\mathcal{G}}$ is determined by Definition 2.1 as Figure 1. Here, each arrowhead in Figure 1 represents the plus direction of each edges.

Let γ be the closed walk $(a, e_{ab}, b, e_{bc}, c, e_{ca}, a)$ on $\mathcal{G}$. We see that $\widetilde{V}^{\gamma}_{\phi,3}(a_1) = \{a_1, a_2, a_3\} = \phi_V^{-1}(a)$, hence we have $\mathrm{NV}_\phi(\gamma, a_1) = \mathfrak{S}_3$. For the walk $\gamma^{(2)} = (c, e_{ca}, a, e_{ab}, b, e_{bc}, c)$, we can see that $\widetilde{V}^{\gamma^{(2)}}_{\phi,3}(c_1) = \{c_1, c_2, c_3, c_5\}$. Hence we obtain

$$\mathrm{NV}_\phi(\gamma^{(2)}, c_1) = \{\mathrm{id}, (1\,3), (1\,2), (1\,3\,2)\}.$$

Therefore, $\mathrm{NV}_\phi(\gamma) \neq \mathrm{NV}_\phi(\gamma^{(2)})$.

Remark 2.14. Let G be a finite abelian group, let $\phi : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a G-cover of a graph $\mathcal{G}$, and let $\gamma = (v_0, e_1^{\sigma_1}, \dots, v_n)$ be a closed walk on $\mathcal{G}$. If $v_i = v_j$ for some $0 \le i < j < n$, then $\gamma^{(i)}$ splits into two closed walks:

$$\gamma^{(i)} = (v_i, e_{i+1}^{\sigma_{i+1}}, \dots, v_j, e_{j+1}^{\sigma_{j+1}}, \dots, v_n, e_1^{\sigma_1}, v_1, \dots, v_i) = \gamma_1\gamma_2,$$

where $\gamma_1 = (v_i, e_{i+1}^{\sigma_{i+1}}, \dots, v_j)$ and $\gamma_2 = (v_j, e_{j+1}^{\sigma_{j+1}}, \dots, v_n, e_1^{\sigma_1}, v_1, \dots, v_i)$. By Lemma 2.11 and 2.12, we have

$$\mathrm{NV}_\phi(\gamma) = \mathrm{NV}_\phi(\gamma_1)\mathrm{NV}_\phi(\gamma_2).$$

Thus, in the case where G is abelian, it is enough to compute $\mathrm{NV}_\phi(\gamma)$ for simple closed walks γ in order to compute net voltage classes for all closed walks on $\mathcal{G}$.

3. Splitting graphs and embedded topology

In this section, we define the *splitting graph* of plane curves for a Galois cover of $\mathbb{P}^2$, and study a relation between the splitting graph and the embedded topology of plane curves. Here a Galois cover of $\mathbb{P}^2$ is a finite morphism $\phi : X \to \mathbb{P}^2$ such that X is normal and the extension of rational function fields $\mathbb{C}(X)/\mathbb{C}(\mathbb{P}^2)$ is Galois. In the first subsection, we consider the splitting graph for general Galois covers. In the second subsection, we give a method of computing net voltage classes for the splitting graph in the case of cyclic covers with a certain assumption.

3.1. **Splitting graphs for Galois covers.** We prepare some notation in order to define the splitting graph. Let Y be a normal surface, let $\mathcal{C}$ be a reduced Weil divisor on Y, and let $P \in \mathcal{C}$ be a singular point of $\mathcal{C}$ such that Y is smooth at P. We define $\mathrm{Irr}(\mathcal{C})$ and $\mathrm{LB}_P(\mathcal{C})$ as the sets of irreducible components of $\mathcal{C}$ and local branches of $\mathcal{C}$ at P, respectively. Here a *local branch* of $\mathcal{C}$ at P is an irreducible component of the germ $(\mathcal{C}, P)$. For $C \in \mathrm{Irr}(\mathcal{C})$ and $b \in \mathrm{LB}_P(\mathcal{C})$, we say that C *contains* b, denote by $b \subset C$, if b is an irreducible component of the germ (C, P). By abuse of notation, for $b \in \mathrm{LB}_P(\mathcal{C})$ and a morphism $\varphi : Y \to Y'$ with Y' smooth at $\varphi(P)$, let $\varphi(b)$ denote the image of b under the morphism $\varphi_P|_{(\mathcal{C},P)} : (\mathcal{C}, P) \to (Y', \varphi(P))$ of germs. For a reduced Weil divisor $\mathcal{B}$ on Y with $\mathrm{Sing}(Y) \subset \mathcal{B}$, put $\mathrm{Sing}(\mathcal{C} \setminus \mathcal{B}) := \{P \in \mathrm{Sing}(\mathcal{C}) \mid P \notin \mathcal{B}\}$.

Definition 3.1. Let G be a finite group, and let $\phi : X \to \mathbb{P}^2$ be a G-cover of $\mathbb{P}^2$. Let $\mathcal{B}_\phi$ be the branch locus of ϕ, and let $\mathcal{C} \subset \mathbb{P}^2$ be a plane curve such that $\mathcal{C} \cap \mathcal{B}_\phi$ is finite (equivalently, $\mathcal{C}$ and $\mathcal{B}_\phi$ have no common components).

(i) The *incidence graph of* $\mathcal{C}$ *with respect to* ϕ is the bipartite graph $\mathcal{G} = \mathcal{G}_{\phi,\mathcal{C}}$ such that the set $V_\mathcal{G}$ of vertices has the partition

$(V_{\mathcal{G},0}, V_{\mathcal{G},1})$, where

$$V_{\mathcal{G},0} := \{v_P \mid P \in \mathrm{Sing}(\mathcal{C} \setminus \mathcal{B}_\phi)\} \quad \text{and}$$
$$V_{\mathcal{G},1} := \{v_C \mid C \in \mathrm{Irr}(\mathcal{C})\};$$

and the set $E_{\mathcal{G}}$ of edges is

$$E_{\mathcal{G}} := \bigcup_{P \in \mathrm{Sing}(\mathcal{C} \setminus \mathcal{B}_\phi)} \{e_{P,b} \mid b \in \mathrm{LB}_P(\mathcal{C})\},$$

where we define the initial and terminal vertices of the plus direction $e^+_{P,b}$ of $e_{P,b} \in E_{\mathcal{G}}$ by

$$e^+_{P,b}(0) := v_P \in V_{\mathcal{G},0}, \qquad e^+_{P,b}(1) := v_C \in V_{\mathcal{G},1}$$

for the irreducible component $C \in \mathrm{Irr}(\mathcal{C})$ with $b \subset C$.

(ii) The *splitting graph of $\mathcal{C}$ for ϕ* is the graph $\mathcal{S} := \mathcal{S}_{\phi,\mathcal{C}}$ with the action of G satisfying the following conditions;

(ii-a) the set $V_{\mathcal{S}}$ has the partition $(\widetilde{V}_{\mathcal{S},0}, \widetilde{V}_{\mathcal{S},1})$, where

$$\widetilde{V}_{\mathcal{S},0} := \bigcup_{P \in \mathrm{Sing}(\mathcal{C} \setminus \mathcal{B}_\phi)} \left\{ v_{\widetilde{P}} \,\middle|\, \widetilde{P} \in \phi^{-1}(P) \right\},$$
$$\widetilde{V}_{\mathcal{S},1} := \left\{ v_{\widetilde{C}} \,\middle|\, \widetilde{C} \in \mathrm{Irr}(\phi^*\mathcal{C}) \right\};$$

(ii-b) the set $E_{\mathcal{S}}$ of edges is

$$E_{\mathcal{S}} := \bigcup_{\widetilde{P} \in \phi^{-1}(\mathrm{Sing}(\mathcal{C} \setminus \mathcal{B}_\phi))} \left\{ e_{\widetilde{P},\tilde{b}} \,\middle|\, \tilde{b} \in \mathrm{LB}_{\widetilde{P}}(\phi^*\mathcal{C}) \right\},$$

where $e^+_{\widetilde{P},\tilde{b}}(0) := v_{\widetilde{P}} \in \widetilde{V}_{\mathcal{S},0}$ and $e^+_{\widetilde{P},\tilde{b}}(1) := v_{\widetilde{C}} \in \widetilde{V}_{\mathcal{S},1}$ for $\widetilde{C} \in \mathrm{Irr}(\phi^*\mathcal{C})$ with $\tilde{b} \subset \widetilde{C}$;

(ii-c) G acts on $\mathcal{S}$ via the image under the covering transformation $g : X \to X$ for each $g \in G$, i.e., for $v_{\tilde{x}} \in V_{\mathcal{S}}$, $e_{\widetilde{P},\tilde{b}} \in E_{\mathcal{S}}$ and $g \in G$, we define $g \cdot v_{\tilde{x}} := v_{g(\tilde{x})}$ and $g \cdot e_{\widetilde{P},\tilde{b}} := e_{g(\widetilde{P}),g(\tilde{b})}$ (see Remark 3.2).

Remark 3.2. We define the action of G on $\widetilde{V}_{\mathcal{S},1}$ by using the image $\widetilde{C} \mapsto g(\widetilde{C})$ for $g \in G$ and $\widetilde{C} \in \mathrm{Irr}(\phi^*\mathcal{C})$, NOT the pull-back $\widetilde{C} \mapsto g^*\widetilde{C}$. Since $\tilde{b} \subset \widetilde{C}$ if and only if $g(\tilde{b}) \subset g(\widetilde{C})$ for $\tilde{b} \in \mathrm{LB}_{\widetilde{P}}(\phi^*\mathcal{C})$ ($\widetilde{P} \in \phi^{-1}(\mathrm{Sing}(\mathcal{C} \setminus \mathcal{B}_\phi))$) and $\widetilde{C} \in \mathrm{Irr}(\phi^*\mathcal{C})$, the endpoint sets satisfy $g \cdot (V_{\mathcal{S}}(e_{\widetilde{P},\tilde{b}})) = V_{\mathcal{S}}(g \cdot e_{\widetilde{P},\tilde{b}})$, hence G acts on $\mathcal{S}$ by Definition 3.1 (ii-c).

Lemma 3.3. *Let $\phi : X \to \mathbb{P}^2$ be a G-cover, and let $\mathcal{C} \subset \mathbb{P}^2$ be a plane curve such that $\mathcal{C} \cap \mathcal{B}_\phi$ is finite. Put the incidence graph $\mathcal{G} := \mathcal{G}_{\phi,\mathcal{C}}$ and* *the splitting graph* $\mathcal{S} := \mathcal{S}_{\phi,\mathcal{C}}$. *Let* $\phi_{\mathcal{C}} = (\phi_{\mathcal{C},V}, \phi_{\mathcal{C},E}) : \mathcal{S} \to \mathcal{G}$ *be the map defined by*

$$\phi_{\mathcal{C},V}(v_{\tilde{x}}) := v_{\phi(\tilde{x})}, \qquad \phi_{\mathcal{C},E}(e_{\widetilde{P},\tilde{b}}) := e_{\phi(\widetilde{P}),\phi(\tilde{b})}$$

for $\tilde{x} \in \phi^{-1}(\mathrm{Sing}(\mathcal{C} \setminus \mathcal{B}_{\phi})) \cup \mathrm{Irr}(\phi^{*}\mathcal{C})$, $\widetilde{P} \in \phi^{-1}(\mathrm{Sing}(\mathcal{C} \setminus \mathcal{B}_{\phi}))$ *and* $\tilde{b} \in \mathrm{LB}_{\widetilde{P}}(\phi^{*}\mathcal{C})$. *Then* $\phi_{\mathcal{C}}$ *is a* G*-cover of graphs.*

Proof. Let $e_{\widetilde{P},\tilde{b}}$ be an edge of $\mathcal{S}$. Its endpointset is $V_{\mathcal{S}}(e_{\widetilde{P},\tilde{b}}) = \{v_{\widetilde{P}}, v_{\widetilde{C}}\}$ for $\widetilde{C} \in \mathrm{Irr}(\phi^{*}\mathcal{C})$ with $\tilde{b} \subset \widetilde{C}$. Put $P := \phi(\widetilde{P})$, $C := \phi(\widetilde{C})$ and $b := \phi(\tilde{b})$ the local branch of $\mathcal{C}$ at P. Since $b \subset C$, we have $V_{\mathcal{G}}(e_{P,b}) = \{v_P, v_C\}$. Thus $\phi_{\mathcal{C},V}(V_{\mathcal{S}}(e_{\widetilde{P},\tilde{b}})) = V_{\mathcal{G}}(\phi_{\mathcal{C},E}(e_{\widetilde{P},\tilde{b}}))$, and $\phi_{\mathcal{C}}$ is a map of graphs.

The group G acts transitively on both of $\mathrm{Irr}(\phi^{*}C)$ and $\phi^{-1}(P)$ for $C \in \mathrm{Irr}(\mathcal{C})$ and $P \in \mathrm{Sing}(\mathcal{C} \setminus \mathcal{B}_{\phi})$. Hence G acts transitively on both of $\phi_V^{-1}(v_C)$ and $\phi_V^{-1}(v_P)$, and we have $V_{\mathcal{G}} = V_{\mathcal{S}}/G$. Since $P \in \mathrm{Sing}(\mathcal{C} \setminus \mathcal{B}_{\phi})$ is not a branch point of ϕ, G acts transitively and freely on the set

$$\{\tilde{b} \in \mathrm{LB}_{\widetilde{P}}(\phi^{*}\mathcal{C}) \mid \widetilde{P} \in \phi^{-1}(P), \phi(\tilde{b}) = b\}$$

for $b \in \mathrm{LB}_P(\mathcal{C})$. Thus G acts transitively and freely on $\phi_E^{-1}(e_{P,b})$ for $P \in \mathrm{Sing}(\mathcal{C} \setminus \mathcal{B}_{\phi})$ and $b \in \mathrm{LB}_P(\mathcal{C})$, and we obtain $E_{\mathcal{G}} = E_{\mathcal{S}}/G$. Therefore, ϕ is a G-cover of graphs. $\square$

Let G be a finite group, and let $\mathcal{B} \subset \mathbb{P}^2$ be a plane curve. It is known that a surjective homomorphism $\rho : \pi_1(\mathbb{P}^2 \setminus \mathcal{B}) \twoheadrightarrow G$ induces a G-cover $\phi : X \to \mathbb{P}^2$, uniquely up to isomorphism over $\mathbb{P}^2$, whose branch locus is contained in $\mathcal{B}$. Here $\pi_1(\mathbb{P}^2 \setminus \mathcal{B})$ is the fundamental group of $\mathbb{P}^2 \setminus \mathcal{B}$. Conversely, a G-cover $\phi : X \to \mathbb{P}^2$ branched at $\mathcal{B}$ induces a surjective homomorphism $\rho : \pi_1(\mathbb{P}^2 \setminus \mathcal{B}) \twoheadrightarrow G$. We roughly recall the surjection $\rho : \pi_1(\mathbb{P}^2 \setminus \mathcal{B}) \twoheadrightarrow G$ induced by a G-cover $\phi : X \to \mathbb{P}^2$ (cf. [13] for details).

Let $\phi : X \to \mathbb{P}^2$ be a G-cover branched at $\mathcal{B}$, and we fix a base point $P_0 \in \mathbb{P}^2 \setminus \mathcal{B}$. Put $U := \mathbb{P}^2 \setminus \mathcal{B}$ and $\widetilde{U} := X \setminus \phi^{-1}(\mathcal{B})$. Any element $[\gamma]$ of $\pi_1(U) = \pi_1(U, P_0)$ can be represented by a closed path $\gamma : [0,1] \to U$ with $\gamma(0) = \gamma(1) = P_0$. For a point $\widetilde{P} \in \widetilde{U}$, put $P := \phi(\widetilde{P})$. Let $p : [0,1] \to U$ be a path from P to P_0. Then the closed path $p^{-1}\gamma p$ is uniquely lifted in a path $\lambda : [0,1] \to \widetilde{U}$ with $\lambda(0) = \widetilde{P}$. It is known that the point $\lambda(1)$ depends only on the choice of $[\gamma]$ and $\widetilde{P}$, and that $[\gamma]$ gives an isomorphism $g_{\gamma} : \widetilde{U} \to \widetilde{U}$ defined by $g_{\gamma}(\widetilde{P}) = \lambda(1)$. Then this correspondence gives a surjective homomorphism $\pi_1(U) \to \mathrm{Aut}_{\phi}(\widetilde{U})$ defined by $[\gamma] \mapsto g_{\gamma}$, where $\mathrm{Aut}_{\phi}(\widetilde{U})$ is the group $\{g \in \mathrm{Aut}(\widetilde{U}) \mid \phi \circ g = \phi\}$. Since $\mathrm{Aut}_{\phi}(\widetilde{U})$ is isomorphic to G, we obtain a surjective homomorphism $\rho : \pi_1(U) \to G$.

Theorem 3.4. *Let G be a finite group, and let $\mathcal{B}_i$ $(i = 1, 2)$ be two plane curves such that there are surjections $\rho_i : \pi_1(\mathbb{P}^2 \setminus \mathcal{B}_i) \twoheadrightarrow G$. For each $i = 1, 2$, let $\phi_i : X_i \to \mathbb{P}^2$ be a G-cover branched at $\mathcal{B}_i$ induced by ρ_i, and let $\mathcal{C}_i$ be a plane curve such that $\mathcal{C}_i \cap \mathcal{B}_i$ is finite. Assume that there exists a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h(\mathcal{B}_1) = \mathcal{B}_2$, $h(\mathcal{C}_1) = \mathcal{C}_2$ and $\rho_2 \circ h_* = \tau \circ \rho_1$ for some automorphism $\tau : G \to G$, where $h_* : \pi_1(\mathbb{P}^2 \setminus \mathcal{B}_1) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2)$ is the isomorphism induced by h. Put $\mathcal{G}_i := \mathcal{G}_{\phi_i, \mathcal{C}_i}$ and $\mathcal{S}_i := \mathcal{S}_{\phi_i, \mathcal{C}_i}$. Then the following statements hold:*

(i) *h induces an isomorphism $\theta_h : \mathcal{G}_1 \to \mathcal{G}_2$ of the incidence graphs preserving the partitions, i.e., $\theta_{h,V}(V_{\mathcal{G}_1,j}) = V_{\mathcal{G}_2,j}$ for $j = 0, 1$, where $(V_{\mathcal{G}_i,0}, V_{\mathcal{G}_i,1})$ is the partition of $V_{\mathcal{G}_i}$ in Definition 3.1 (i);*

(ii) *the splitting graphs $\mathcal{S}_1$ and $\mathcal{S}_2$ are (θ_h, τ)-equivalent, $\mathcal{S}_1 \sim_{(\theta_h, \tau)} \mathcal{S}_2$.*

Proof. We define $\theta_{h,V} : V_{\mathcal{G}_1} \to V_{\mathcal{G}_2}$ and $\theta_{h,E} : E_{\mathcal{G}_1} \to E_{\mathcal{G}_2}$ by

$$\theta_{h,V}(v_{x_1}) := v_{h(x_1)}, \qquad \theta_{h,E}(e_{P_1,b_1}) := e_{h(P_1),h(b_1)},$$

respectively, for $x_1 \in \mathrm{Sing}(\mathcal{C}_1 \setminus \mathcal{B}_1) \cup \mathrm{Irr}(\mathcal{C}_1)$, $P_1 \in \mathrm{Sing}(\mathcal{C}_1 \setminus \mathcal{B}_1)$ and $b_1 \in \mathrm{LB}_{P_1}(\mathcal{C}_1)$. Since $h(\mathcal{B}_1) = \mathcal{B}_2$ and $h(\mathcal{C}_1) = \mathcal{C}_2$, $\theta_{h,V}$ is well-defined, bijective and preserving the partitions. The map $\theta_{h,E}$ is well-defined and bijective since $b_1 \in \mathrm{LB}_{P_1}(\mathcal{C}_1)$ if and only if $h(b_1) \in \mathrm{LB}_{h(P_1)}(\mathcal{C}_2)$. Moreover, we have $V_{\mathcal{G}_2}(\theta_{h,E}(e_{P_1,b_1})) = \theta_{h,V}(V_{\mathcal{G}_1}(e_{P_1,b_1}))$ since $b_1 \subset C_1$ if and only if $h(b_1) \subset h(C_1)$ for $C_1 \in \mathrm{Irr}(\mathcal{C}_1)$. Therefore, $\theta_h = (\theta_{h,V}, \theta_{h,E})$ is an isomorphism of the incidence graphs $\mathcal{G}_i$, and assertion (i) holds.

We denote $\mathbb{P}^2 \setminus \mathcal{B}_i$ and $X_i \setminus \phi_i^{-1}(\mathcal{B}_i)$ by U_i and $\widetilde{U}_i$, respectively, for each $i = 1, 2$. By the uniqueness of unramified G-covers induced by ρ_i, there is a homeomorphism $\tilde{h} : \widetilde{U}_1 \to \widetilde{U}_2$ satisfying $h|_{U_1} \circ \phi_1|_{\widetilde{U}_1} = \phi_2|_{\widetilde{U}_2} \circ \tilde{h}$. Hence we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{U}_1 & \xrightarrow{\tilde{h}} & \widetilde{U}_2 \\ {\scriptstyle \phi_1|_{\widetilde{U}_1}} \downarrow & & \downarrow {\scriptstyle \phi_2|_{\widetilde{U}_2}} \\ U_1 & \xrightarrow{h|_{U_1}} & U_2 \end{array}$$

By the same argument of proof of (i), we have an isomorphism $\tilde{\theta}_h : \mathcal{S}_1 \to \mathcal{S}_2$ of graphs. From the above diagram, it is easy to see that $\theta_{h,\bullet} \circ \phi_{\mathcal{C}_1,\bullet} = \phi_{\mathcal{C}_2,\bullet} \circ \tilde{\theta}_{h,\bullet}$ for each $\bullet = V, E$. Thus Condition (i) in Definition 2.3 holds.

Take a point $\widetilde{P}_1 \in \widetilde{U}_1$. Let $\gamma : [0, 1] \to U_1$ be a closed path with $\gamma(0) = \phi_1(\widetilde{P}_1)$, and let $\tilde{\gamma} : [0, 1] \to \widetilde{U}_1$ be the lift of γ satisfying $\tilde{\gamma}(0) = \widetilde{P}_1$. Then $g_\gamma \cdot \widetilde{P}_1 = \widetilde{Q}_1 := \tilde{\gamma}(1)$, where $g_\gamma = \rho_1([\gamma])$. Note that $h_*([\gamma])$

is the element of $\pi_1(U_2)$ represented by the path $h \circ \gamma : [0,1] \to U_2$, and that $\tilde{h} \circ \tilde{\gamma} : [0,1] \to \widetilde{U}_2$ is the lift of $h \circ \gamma$ whose initial point is $\widetilde{P}_2 := \tilde{h}(\widetilde{P}_1)$, i.e., $\tilde{h} \circ \tilde{\gamma}(0) = \widetilde{P}_2$. Hence we have $g_{h\circ\gamma} \cdot \widetilde{P}_2 = \tilde{h}(\widetilde{Q}_1)$, where $g_{h\circ\gamma} = \rho_2([h \circ \gamma])$. Since $\rho_2 \circ h_* = \tau \circ \rho_1$ by the assumption, we have

$$g_{h\circ\gamma} = \rho_2 \circ h_*([\gamma]) = \tau \circ \rho_1([\gamma]) = \tau(g_\gamma).$$

Hence we obtain

$$\tilde{h}(g_\gamma \cdot \widetilde{P}_1) = \tilde{h}(\widetilde{Q}_1) = g_{h\circ\gamma} \cdot \widetilde{P}_2 = \tau(g_\gamma) \cdot \tilde{h}(\widetilde{P}_1)$$

for any $\widetilde{P}_1 \in \widetilde{U}_1$ and any closed path γ with $\gamma(0) = \phi_1(\widetilde{P}_1)$. By Condition (ii-c) in Definition 3.1, we have $\tilde{\theta}_{h,V}(g \cdot \tilde{v}) = \tau(g) \cdot \tilde{\theta}_{h,V}(\tilde{v})$ and $\tilde{\theta}_{h,E}(g \cdot \tilde{e}) = \tau(g) \cdot \tilde{\theta}_{h,E}(\tilde{e})$ for any $\tilde{v} \in V_{\mathcal{S}_1}$, $\tilde{e} \in E_{\mathcal{S}_1}$ and $g \in G$. Therefore we conclude that $\mathcal{S}_1 \sim_{(\theta_h,\tau)} \mathcal{S}_2$. □

Remark 3.5. Let $\mathcal{C}_i + \mathcal{B}_i$ $(i = 1, 2)$ be two plane curves, let $\phi_i : X_i \to \mathbb{P}^2$ be G-covers, and put $\mathcal{G}_i := \mathcal{G}_{\phi_i,\mathcal{C}_i}$ and $\mathcal{S}_i := \mathcal{S}_{\phi_i,\mathcal{C}_i}$ as in Theorem 3.4.

(i) If there exist tubular neighborhoods $\mathcal{T}(\mathcal{C}_i + \mathcal{B}_i) \subset \mathbb{P}^2$ of $\mathcal{C}_i + \mathcal{B}_i$ $(i = 1, 2)$ such that there is a homeomorphism $h' : \mathcal{T}(\mathcal{C}_1 + \mathcal{B}_1) \to \mathcal{T}(\mathcal{C}_2 + \mathcal{B}_2)$ with $h'(\mathcal{C}_1) = \mathcal{C}_2$ and $h'(\mathcal{B}_1) = \mathcal{B}_2$, then h' induces an isomorphism $\theta_{h'} : \mathcal{G}_1 \to \mathcal{G}_2$ preserving the partitions as the proof of Theorem 3.4 (i).

(ii) In the case where $\mathcal{C}_i$ are irreducible, the cardinality of $\widetilde{V}_{\mathcal{S}_i,1} \subset V_{\mathcal{S}_i}$ is equal to the splitting number $s_{\phi_i}(\mathcal{C}_i)$, which is the number of irreducible components of $\phi^*\mathcal{C}_i$. Hence, if $s_{\phi_1}(\mathcal{C}_1) \neq s_{\phi_2}(\mathcal{C}_2)$, then there is no isomorphism $\theta : \mathcal{G}_1 \to \mathcal{G}_2$ preserving the partitions such that $\mathcal{S}_1 \sim_{(\theta,\tau)} \mathcal{S}_2$ for any automorphism $\tau : G \to G$. Since the splitting number is not determined by the fundamental group by [18], the splitting graph is not determined by the one.

(iii) It is easy to see that the number of connected components of $\mathcal{S}_i$ is equal to the connected number $c_{\phi_i}(\mathcal{C}_i)$ (cf. [19]).

In order to restrict the possibility of $\tau : G \to G$ in Theorem 3.4, we discuss the image of a meridian of an irreducible component of a plane curve $\mathcal{B}$ under a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$. Let $P \in \mathcal{B}$ be a smooth point of $\mathcal{B}$. Take an open neighborhood $U \subset \mathbb{P}^2$ of P and a system of local coordinates (x', y') of U with $P = (0, 0)$ so that $\mathcal{B}$ is defined by $x' = 0$. Let δ_ϵ be the closed path $[0, 1] \to U$ defined by $t \mapsto (\epsilon \exp(2\pi\sqrt{-1}\, t), 0)$ for a small number $\epsilon > 0$. We call $m := p\delta_\epsilon p^{-1}$ a *meridian* of $\mathcal{B}$ at P, where $p : [0, 1] \to \mathbb{P}^2$ is a path from the base point $* \in \mathbb{P}^2 \setminus \mathcal{B}$ to the point $(\epsilon, 0) \in U$. By abuse of notation, the class $[m] \in \pi_1(\mathbb{P}^2 \setminus \mathcal{B})$ be also called the *meridian* of $\mathcal{B}$ at P. Note that, for a meridian $[m]$ of $\mathcal{B}$ at P, any conjugate of $[m]$ is also a meridian at P, and that a meridian $[m']$ at P' is a conjugate of $[m]$ in $\pi_1(\mathbb{P}^2\setminus\mathcal{B})$ if P and P' are contained in an irreducible component of $\mathcal{B}$. The next lemma is effective to restrict the possibility of a automorphism $\tau : G \to G$ in Theorem 3.4.

Lemma 3.6. *Let $\mathcal{B}_1, \mathcal{B}_2$ be two plane curves, and let $P_1 \in B_1$ be a smooth point of $\mathcal{B}_1$. Assume that there is a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h(\mathcal{B}_1) = \mathcal{B}_2$. Put $P_2 := h(P_1)$. Let $m_i : [0,1] \to \mathbb{P}^2$ be a meridian of $\mathcal{B}_i$ at P_i for each $i = 1, 2$. Then the closed path $h \circ m_1$ is homotpically equivalent to either a certain meridian of $\mathcal{B}_2$ at P_2 or its inverse in $\mathbb{P}^2 \setminus \mathcal{B}_2$. Equivalently, the class $[h \circ m_1]$ is a conjugate of either $[m_2]$ or $[m_2]^{-1}$ in $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_2)$ for any meridian m_2 of $\mathcal{B}_2$ at P_2.*

Proof. Fix base points $*_1 \in \mathbb{P}^2\setminus\mathcal{B}_1$ and $*_2 := h(*_1)$ of $\mathbb{P}^2\setminus\mathcal{B}_1$ and $\mathbb{P}^2\setminus\mathcal{B}_2$, respectively. For each $i = 1, 2$, let $U_i \subset \mathbb{P}^2$ be an open neighborhood of P_i, and let (x'_i, y'_i) be a system of local coordinates of U_i with $P_i = (0, 0)$ so that $\mathcal{B}_i$ is defined by $x'_i = 0$. Let $\epsilon_2 > 0$ be a small number such that $m_2 = p_2\delta_{\epsilon_2}p_2^{-1}$, where δ_{ϵ_2} is the closed path in U_2, and p_2 is a path from $*_2$ to $(\epsilon_2, 0) \in U_2$ in $\mathbb{P}^2 \setminus \mathcal{B}_2$. For a positive number $\epsilon > 0$, put

$$D_{i,\epsilon} := \{(x'_i, y'_i) \mid |x'_i|^2 + |y'_i|^2 \leq \epsilon^2\} \subset U_i.$$

Since the preimage $h^{-1}(\mathrm{Int}(D_{2,\epsilon_2}))$ of the interior $\mathrm{Int}(D_{2,\epsilon_2})$ of D_{2,ϵ_2} is open in U_1, there exists a positive number $\epsilon_1 > 0$ such that $D_{1,\epsilon_1} \subset h^{-1}(\mathrm{Int}(D_{2,\epsilon_2}))$. We may assume that δ_{ϵ_1} with respect to (x'_1, y'_1) satisfies that $m_1 = p_1\delta_{\epsilon_1}p_1^{-1}$, where p_1 is a path from the base point $*_1$ to $(\epsilon_1, 0) \in U_1$ in $\mathbb{P}^2 \setminus \mathcal{B}_1$. Since $h(\mathrm{Int}(D_{1,\epsilon_1}))$ is open in U_2, there exists a positive number $\epsilon'_2 < \epsilon_2$ such that $D_{2,\epsilon'_2} \subset h(\mathrm{Int}(D_{1,\epsilon_1}))$. Since $h(\mathcal{B}_1) = \mathcal{B}_2$, the inclusions $D_{2,\epsilon'_2} \to D_{2,\epsilon_2}$, $D_{2,\epsilon'_2} \to h(D_{1,\epsilon_1})$ and $h(D_{1,\epsilon_1}) \to D_{2,\epsilon_2}$ induce the morphisms

$$\begin{aligned} &i_{1*} : \pi_1(D_{2,\epsilon'_2} \setminus \mathcal{B}_2) \to \pi_1(D_{2,\epsilon_2} \setminus \mathcal{B}_2), \\ &i_{2*} : \pi_1(D_{2,\epsilon'_2} \setminus \mathcal{B}_2) \to \pi_1(h(D_{1,\epsilon_1} \setminus \mathcal{B}_1)) \ \text{ and} \\ &i_{3*} : \pi_1(h(D_{1,\epsilon_1} \setminus \mathcal{B}_1)) \to \pi_1(D_{2,\epsilon_2} \setminus \mathcal{B}_2), \end{aligned}$$

respectively. Note that we have

$$\pi_1(D_{2,\epsilon'_2} \setminus \mathcal{B}_2) \cong \pi_1(D_{2,\epsilon_2} \setminus \mathcal{B}_2) \cong \pi_1(D_{1,\epsilon_1} \setminus \mathcal{B}_1) \cong \mathbb{Z}.$$

Since $i_{1*} = i_{3*} \circ i_{2*}$ and i_{1*} is isomorphic, the composition of h_* and i_{3*} maps a generator of $\pi_1(D_{1,\epsilon_1} \setminus \mathcal{B}_1)$ to a generator of $\pi_1(D_{2,\epsilon_2} \setminus \mathcal{B}_2)$. Thus $h \circ \delta_{\epsilon_1}$ is homotopically equivalent to either δ_{ϵ_2} or $\delta_{\epsilon_2}^{-1}$ in $\mathbb{P}^2 \setminus \mathcal{B}_2$. This implies that $h_*([m_1])$ is a conjugate of either $[m_2]$ or $[m_2]^{-1}$. $\square$

Remark 3.7. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are smooth plane curves of degree d, then $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}_i) \cong \mathbb{Z}_d$. For a meridian $m_{\mathcal{B}_i}$ of $\mathcal{B}_i$, we assume that the class $[m_{\mathcal{B}_i}] \in \pi_1(\mathbb{P}^2 \setminus \mathcal{B}_i)$ corresponds to the generator $[1] \in \mathbb{Z}_d$. By Lemma 3.6, a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ with $h(\mathcal{B}_1) = \mathcal{B}_2$ induces either $\tau_d^+ : \mathbb{Z}_d \to \mathbb{Z}_d$ or $\tau_d^- : \mathbb{Z}_d \to \mathbb{Z}_d$, where the automorphisms $\tau_d^\pm$ are defined by $\tau_d^\pm([1]) = [\pm 1]$, respectively.

We define an equivalence between splitting graphs as follows.

Definition 3.8. Let G be a finite group, and let $\mathcal{B}_i$ $(i = 1, 2)$ be two plane curves such that there are surjections $\rho_i : \pi_1(\mathbb{P}^2 \setminus \mathcal{B}_i) \to G$. Let $\phi_i : X_i \to \mathbb{P}^2$ be a G-cover induced by ρ_i, and let $\mathcal{C}_i$ be a plane curve such that $\mathcal{C}_i \cap \mathcal{B}_i$ is finite for each $i = 1, 2$. The splitting graphs $\mathcal{S}_{\phi_1,\mathcal{C}_1}$ and $\mathcal{S}_{\phi_2,\mathcal{C}_2}$ are said to be *equivalent*, denoted by $\mathcal{S}_{\phi_1,\mathcal{C}_1} \sim \mathcal{S}_{\phi_2,\mathcal{C}_2}$, if there exist a homeomorphism $h' : \mathcal{T}(\mathcal{C}_1 + \mathcal{B}_1) \to \mathcal{T}(\mathcal{C}_2 + \mathcal{B}_2)$ of tubular neighborhoods $\mathcal{T}(\mathcal{C}_i + \mathcal{B}_i) \subset \mathbb{P}^2$ of $\mathcal{C}_i + \mathcal{B}_i$ and an automorphism $\tau : G \to G$ satisfying

(i) $h'(\mathcal{C}_1) = \mathcal{C}_2$ and $h'(\mathcal{B}_1) = \mathcal{B}_2$;
(ii) for any meridian m_B of any irreducible component $B \subset \mathcal{B}_1$, either $\tau(\rho_1([m_B])) = \rho_2([m_{h'(B)}])$ or $\tau(\rho_1([m_B])) = \rho_2([m_{h'(B)}]^{-1})$ for some meridian $m_{h'(B)}$ of $h'(B)$; and
(iii) $\mathcal{S}_{\phi_1,\mathcal{C}_1} \sim_{(\theta_{h'},\tau)} \mathcal{S}_{\phi_2,\mathcal{C}_2}$ as G-covers of graphs, where $\theta_{h'} : \mathcal{G}_{\phi_1,\mathcal{C}_1} \to \mathcal{G}_{\phi_2,\mathcal{C}_2}$ is the isomorphism in Remark 3.5 (i).

Remark 3.9. A homeomorphism $h' : \mathcal{T}(\mathcal{C}_1 + \mathcal{B}_1) \to \mathcal{T}(\mathcal{C}_2 + \mathcal{B}_2)$ in Definition 3.8 gives a correspondence between the combinatorial data of $\mathcal{C}_1 + \mathcal{B}_1$ and $\mathcal{C}_2 + \mathcal{B}_2$, which consist of the sets of irreducible components, singularities, degrees of components, and configuration of components of $\mathcal{C}_i + \mathcal{B}_i$. Conversely, it is known that a correspondence between the combinatorial data induces a homeomorphism h' between tubular neighborhoods (cf. [2, Remark 3]).

By Theorem 3.4 and Lemma 3.6, we obtain the following lemma.

Corollary 3.10. *Under the assumption of Theorem 3.4, the splitting graphs $\mathcal{S}_{\phi_1,\mathcal{C}_1}$ and $\mathcal{S}_{\phi_2,\mathcal{C}_2}$ are equivalent.*

Example 3.11. Let $\mathcal{B}$ be the conic defined by $z^2 - 4xy = 0$, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two 6-nodal irreducible sextics in [7, Example 6.2] and [7, Example 6.3], respectively. Note that $\mathcal{B}$ is a simple contact conic of both of $\mathcal{C}_1$ and $\mathcal{C}_2$. Let $\phi : X \to \mathbb{P}^2$ be the double cover branched at the conic $\mathcal{B}$. Then $\mathcal{C}_1$ is a splitting curve with respect to ϕ, write $\phi^*\mathcal{C}_1 = \widetilde{\mathcal{C}}_1^+ + \widetilde{\mathcal{C}}_1^-$, and $\phi^{-1}(\mathrm{Sing}(\mathcal{C}_1 \setminus \mathcal{B})) = (\widetilde{\mathcal{C}}_1^+ \cap \widetilde{\mathcal{C}}_1^-) \setminus \phi^{-1}(\mathcal{B})$. On the other hand, $\widetilde{\mathcal{C}}_2 := \phi^*\mathcal{C}_2$ is irreducible. Thus the preimage of the 6 nodes of $\mathcal{C}_2$ are the 12 nodes of $\widetilde{\mathcal{C}}_2$. Hence the splitting graphs

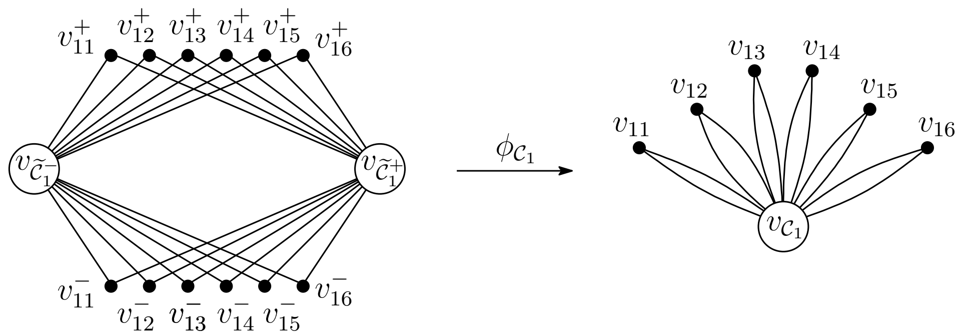

FIGURE 2. The splitting graph of $\mathcal{C}_1$ for ϕ

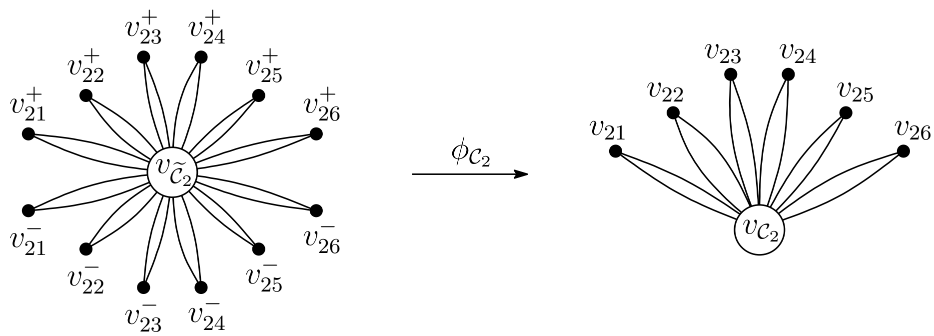

FIGURE 3. The splitting graph of $\mathcal{C}_2$ for ϕ

$\phi_{\mathcal{C}_i} : \mathcal{S}_{\phi,\mathcal{C}_i} \to \mathcal{G}_{\phi,\mathcal{C}_i}$ are as Figure 2 and 3, respectively, since a node consists of two local branches, where v_{ij} are vertices corresponding to the 6 nodes of $\mathcal{C}_i$, and $\phi_{\mathcal{C}_i}^{-1}(v_{ij}) = \{v_{ij}^+, v_{ij}^-\}$. Hence $\mathcal{S}_{\phi_1,\mathcal{C}_1}$ and $\mathcal{S}_{\phi_2,\mathcal{C}_2}$ are not equivalent.

3.2. **Net voltage classes of Splitting graphs for cyclic covers.** Proposition 2.8 and Theorem 3.4 imply that computing net voltage classes of closed walks is effective to distinguish the embedded topology of plane curves. We investigate a computation of net voltage classes for cyclic covers. Let $\overline{\mathcal{B}}$ and $\mathcal{B}$ be divisors

$$\overline{\mathcal{B}} = \sum_{i=1}^{m-1} i \cdot B_i \quad \text{and} \quad \mathcal{B} = \sum_{i=1}^{m-1} B_i$$

on $\mathbb{P}^2$, respectively, where B_i $(i = 1, \dots, m-1)$ are reduced divisors with no common components each other. Note that the degree of $\overline{\mathcal{B}}$ is divisible by m if and only if there exists a surjection $\rho : \pi_1(\mathbb{P}^2 \setminus \mathcal{B}) \twoheadrightarrow \mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ which sends any meridian of $\mathcal{B}$ at any $P_i \in B_i \setminus \mathrm{Sing}(\mathcal{B})$ to the image $[i] \in \mathbb{Z}_m$ of i in $\mathbb{Z}_m$ (cf. [16]). Assume that the degree of $\overline{\mathcal{B}}$ is divisible by m. We call the cyclic cover induced by the surjection $\rho : \pi_1(\mathbb{P}^2 \setminus \mathcal{B}) \twoheadrightarrow \mathbb{Z}_m$ as above the *$\mathbb{Z}_m$-cover of type $\overline{\mathcal{B}}$*.

Let $\phi : X \to \mathbb{P}^2$ be the $\mathbb{Z}_m$-cover of type $\overline{\mathcal{B}}$, and let $\mathcal{C} \subset \mathbb{P}^2$ be a plane curve such that $\mathcal{C} \cap \mathcal{B}$ is finite. We assume the following condition to compute the net voltage class of a closed walk for ϕ:

(1) All irreducible components of $\mathcal{C}$ are smooth.

Remark 3.12. Under Assumption (1), the incidence graph $\mathcal{G}_{\phi,\mathcal{C}}$ has no parallel edge. Hence an edge of $\mathcal{G}_{\phi,\mathcal{C}}$ is identified with a pair (v_P, v_C) of vertices $v_P \in V_{\mathcal{G}_{\phi,\mathcal{C}},0}$ and $v_C \in V_{\mathcal{G}_{\phi,\mathcal{C}},1}$. In this case, we omit edges from sequences representing walks on $\mathcal{G}_{\phi,\mathcal{C}}$. Namely, we represent walks by sequences of vertices only.

Let $L \subset \mathbb{P}^2$ be a line which intersects transversally with $\mathcal{C}$, and is not a component of $\mathcal{B}$. Since L does not pass through singularities of $\mathcal{C}$, it is enough to consider the singular points and irreducible components of $\mathcal{C}$ and $\phi^*\mathcal{C}$ over the affine open set $U' := \mathbb{P}^2 \setminus L$ for computing the net voltage class. Hence we consider the restriction $\phi' : \widetilde{U}' \to U'$ of ϕ to $\widetilde{U}' := X \setminus \phi^{-1}(L)$. We regard the coordinate ring of U' as the polynomial ring $\mathbb{C}[x', y']$. Let $F = 0$ be a defining equation of $\overline{\mathcal{B}}$ on U'. By $L \not\subset \mathcal{B}$ and the proof of [18, Theorem 2.7] (cf. [11, Theorem 2.1]), if $C \in \mathrm{Irr}(\mathcal{C})$ is defined by $f = 0$ on U', and if the splitting number of C for ϕ is s, then there are two polynomials $g, h \in \mathbb{C}[x', y']$ satisfying the following equation:

$$F = fg + h^s.$$

Let γ be the following closed walk on the incidence graph $\mathcal{G}_{\mathcal{C}}$:

$$\gamma = (v_{P_1}, v_{C_1}, v_{P_2}, v_{C_2}, \dots, v_{P_n}, v_{C_n}, v_{P_{n+1}}),$$

where $P_i \in \mathrm{Sing}(\mathcal{C} \setminus \mathcal{B})$ with $P_{n+1} = P_1$ and $C_i \in \mathrm{Irr}(\mathcal{C})$. We fix a defining equation $f_i = 0$ of C_i and polynomials $g_i, h_i \in \mathbb{C}[x', y']$ satisfying

$$F = f_i g_i + h_i^{s_i} \tag{2}$$

for each $i = 1, \dots, n$, where s_i is the splitting number of C_i for ϕ. Since P_{i+1} is an intersection of C_i and C_{i+1} for $i = 1, \dots, n$, we have $F(P_{i+1}) = (h_i(P_{i+1}))^{s_i} = (h_{i+1}(P_{i+1}))^{s_{i+1}}$, where $C_{n+1} := C_1$. For each $i = 1, \dots, n$, we fix a complex number $d_i \in \mathbb{C}$ such that

$$h_i(P_i) = d_i^{\mu_i},$$

where $\mu_i := m/s_i$. Since $(h_i(P_{i+1}))^{s_i} = (h_{i+1}(P_{i+1}))^{s_{i+1}}$, there is an integer α_i with $0 \le \alpha_i < s_i$ such that

$$h_i(P_{i+1}) = (\zeta_m^{\mu_i})^{\alpha_i} d_{i+1}^{\mu_i} \tag{3}$$

for each $i = 1, \dots, n$, where $\zeta_m := \exp(2\pi\sqrt{-1}/m)$ and $d_{n+1} = d_1$. We put $\alpha := \sum_{i=1}^{n} \alpha_i$.

Theorem 3.13. *Under the above circumstance, the following equation holds:*

$$\mathrm{NV}_{\phi}(\gamma) = [\alpha] + s\mathbb{Z}_m := \{[\alpha + sk] \in \mathbb{Z}_m \mid k \in \mathbb{Z}\},$$

where s is the greatest common divisor of $s_1, \ldots, s_n$.

Proof. Let $\widetilde{U}''$ be the subvariety of $U' \times \mathbb{C}$ defined by

$$t^m = F,$$

where t is a coordinate of $\mathbb{C}$, and let $\phi'' : \widetilde{U}'' \to U'$ be the projection. Note that $\widetilde{U}'$ is the normalization of $\widetilde{U}''$. Moreover, the action of $\mathbb{Z}_m$ on $\widetilde{U}''$ is given by

$$[1]\cdot(P, \zeta_m^j d_P) = (P, \zeta_m^{j+1} d_P), \tag{4}$$

where $[1]$ denotes the image of $1 \in \mathbb{Z}$ in $\mathbb{Z}_m$ (cf. Remark 3.14), and d_P is a complex number with $d_P^m = F(P)$. Since $\widetilde{U}''$ is smooth over $U' \setminus \mathcal{B}$, we have $\widetilde{U}' \setminus (\phi')^{-1}(\mathcal{B}) \cong \widetilde{U}'' \setminus (\phi'')^{-1}(\mathcal{B})$. Thus it is enough to consider $(\phi'')^*\mathcal{C}$ to compute the net voltage class.

Since $F(P_i) = d_i^m$ and $\widetilde{U}''$ is defined by $t^m = F$ in $U' \times \mathbb{C}$, the preimage of P_i under $\phi'' : \widetilde{U}'' \to U'$ consists of the following m points

$$\widetilde{P}_{i,j} := (P_i, \zeta_m^j d_i) \in \widetilde{U}'' \subset U' \times \mathbb{C} \quad (j = 0, \ldots, m-1)$$

for $i = 1, \ldots, n+1$. Note that $\widetilde{P}_{n+1,j} = \widetilde{P}_{1,j}$ since $P_{n+1} = P_1$ and $d_{n+1} = d_1$. Let $\widetilde{C}_{i,k}$ be the irreducible component of $(\phi'')^*C_i$ defined by the following equation in $U' \times \mathbb{C}$;

$$\widetilde{C}_{i,k} : t^{\mu_i} - (\zeta_m^{\mu_i})^k h_i = f_i = 0$$

for each $i = 1, \ldots, n$ and $k = 0, \ldots, s_i - 1$.

Claim 1. (i) *$\widetilde{P}_{i,j} \in \widetilde{C}_{i,k}$ if and only if $j \equiv k \pmod{s_i}$.*
(ii) *$\widetilde{P}_{i+1,j} \in \widetilde{C}_{i,k}$ if and only if $j \equiv \alpha_i + k \pmod{s_i}$.*

Proof. The condition $\widetilde{P}_{i,j} \in \widetilde{C}_{i,k}$ is equivalent to $(\zeta_m^{\mu_i})^j d_i^{\mu_i} = (\zeta_m^{\mu_i})^k d_i^{\mu_i}$. Hence $\widetilde{P}_{i,j} \in \widetilde{C}_{i,k}$ if and only if $j \equiv k \pmod{s_i}$.

The condition $\widetilde{P}_{i+1,j} \in \widetilde{C}_{i,k}$ is equivalent to $(\zeta_m^{\mu_i})^j d_{i+1}^{\mu_i} = (\zeta_m^{\mu_i})^{\alpha_i + k} d_{i+1}^{\mu_i}$. Thus $\widetilde{P}_{i+1,j} \in \widetilde{C}_{i,k}$ if and only if $j \equiv \alpha_i + k \pmod{s_i}$ □

By Claim 1 (i), we have

$$\{v_{\widetilde{C}_{i,k}} \mid \widetilde{P}_{i,j} \in \widetilde{C}_{i,k}\} = \{v_{\widetilde{C}_{i,k}} \mid k = j + c_i s_i \text{ for some } c_i \in \mathbb{Z}\} \subset V_{2i-1}^{\gamma}(v_{\widetilde{P}_{1,0}})$$

for each $v_{\widetilde{P}_{i,j}} \in V^{\gamma}_{2i-2}(v_{\widetilde{P}_{1,0}})$. By Claim 1 (ii), hence, we obtain the following equation;

$$\begin{aligned} V^{\gamma}_{2i}(v_{\widetilde{P}_{1,0}}) &= \bigcup_{v_{\widetilde{P}_{i,j}} \in V^{\gamma}_{2i-2}(v_{\widetilde{P}_{1,0}})} \left\{ v_{\widetilde{P}_{i+1,j'}} \,\middle|\, j' = j + \alpha_i + b_i s_i \text{ for some } b_i \in \mathbb{Z} \right\} \\ &= \left\{ v_{\widetilde{P}_{i+1,j}} \,\middle|\, j = \sum_{i'=1}^{i} \alpha_{i'} + \sum_{i'=1}^{i} b_{i'} s_{i'} \text{ for some } b_1, \dots, b_i \in \mathbb{Z} \right\} \end{aligned}$$

Since s is the greatest common divisor of $s_1, \dots, s_n$, we obtain the assertion. □

Remark 3.14. Action (4) in the proof of Theorem 3.13 coincides with the monodromy action on $\widetilde{U}'' \setminus (\phi'')^{-1}(\mathcal{B}) \cong \widetilde{U}' \setminus (\phi')^{-1}(\mathcal{B})$ of a meridian $[m_1] \in \pi_1(\mathbb{P}^2 \setminus \mathcal{B})$ at a point $Q_1 \in B_1 \setminus \mathrm{Sing}(\mathcal{B})$. Indeed, the path

$$[0,1] \ni t \mapsto \big(\epsilon \exp(2\pi\sqrt{-1}\, t),\ 0,\ \epsilon^{1/m} \exp(2\pi\sqrt{-1}\, t/m)\big) \in \widetilde{U}'' \setminus (\phi'')^{-1}(\mathcal{B})$$

from $(\epsilon, 0, \epsilon^{1/m})$ to $(\epsilon, 0, \zeta_m \epsilon^{1/m})$ is a lift of the path

$$\delta_\epsilon : [0,1] \ni t \mapsto (\epsilon \exp(2\pi\sqrt{-1}\, t), 0) \in U' \setminus \mathcal{B},$$

where (x', y') is a system of local coordinates of $\mathbb{P}^2$ at Q_1 so that $F = x'$ at Q_1, and $\zeta_m := \exp(2\pi\sqrt{-1}/m)$.

4. Artal arrangements of degree b

In [1], Artal studied plane curves $\mathcal{C} = E + L_1 + L_2 + L_3$, where E is a smooth cubic, and L_i $(i = 1, 2, 3)$ are non-concurrent inflectional tangents of E. He proved that a pair $(\mathcal{C}_1, \mathcal{C}_2)$ of such curves $\mathcal{C}_1, \mathcal{C}_2$ is a Zariski pair if the three tangent points of $\mathcal{C}_1$ are collinear, and those of $\mathcal{C}_2$ are not collinear. Tokunaga proved the same result by a different way in [23]. In [6], such plane curves are called Artal arrangements. In [19], the author defined an Artal arrangement of degree $b \geq 3$ as a plane curve consisting of one smooth curve of degree b and non-concurrent three of its total inflectional tangents. In [19], he partially distinguished the embedded topology of Artal arrangements. In this section, we define Artal arrangements of type $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ for three partitions $\mathfrak{p}_i$ of an integer $d \geq 3$, which is a generalization of Artal arrangements defined in [6] and [19].

Definition 4.1. Let $B \subset \mathbb{P}^2$ be a smooth curve of degree $d \geq 3$.

(i) For a partition $\mathfrak{p} = (e_1, \dots, e_n)$ of d, we call a line $L \subset \mathbb{P}^2$ a *tangent of type* $\mathfrak{p}$ of B if L intersects with B at just n points $P_1, \dots, P_n$ with multiplicity $e_1, \dots, e_n$, respectively.

(ii) Let $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3$ be three partitions of d, and assume that there is a tangent L_i of type $\mathfrak{p}_i$ of B for each $i = 1, 2, 3$. We call the plane curve $B + L_1 + L_2 + L_3$ an *Artal arrangement of type* $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ if $L_1 \cap L_2 \cap L_3 = \emptyset$ and $B \cap L_i \cap L_j = \emptyset$ for any $i \neq j$.

(iii) For an Artal arrangement $\mathcal{A} := B + L_1 + L_2 + L_3$, let $\phi_{\mathcal{A}} : X_{\mathcal{A}} \to \mathbb{P}^2$ be the $\mathbb{Z}_d$-cover of type B. We call $\phi_{\mathcal{A}}$ the *cyclic cover of the Artal arrangement* $\mathcal{A}$. We fix the surjection $\rho_{\mathcal{A}} : \pi_1(\mathbb{P}^2 \setminus B) \twoheadrightarrow \mathbb{Z}_d$ defined by $[m_B] \mapsto [1]$, where m_B is a meridian of B at a point of B. Furthermore, we call the splitting graph $\mathcal{S}_{\phi_{\mathcal{A}}, L_1+L_2+L_3}$ the *splitting graph of* $\mathcal{A}$, and denote it by $\mathcal{S}_{\mathcal{A}}$.

Remark 4.2. For an element σ of the symmetric group $\mathfrak{S}_3$ of three letters, two Artal arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ of type $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ and $(\mathfrak{p}_{\sigma(1)}, \mathfrak{p}_{\sigma(2)}, \mathfrak{p}_{\sigma(3)})$, respectively, have the same combinatorics. To avoid confusion, we introduce an order on the set of partitions as follows:

Let $\mathfrak{p}_i = (e_{i,1}, \ldots, e_{i,n_i})$ $(i = 1, 2)$ be two partitions of d with $1 \leq e_{i,j} \leq e_{i,j'}$ for $j < j'$. Assume that $\mathfrak{p}_1 \neq \mathfrak{p}_2$, and put $j_0 := \min\{j \mid e_{1,j} \neq e_{2,j}\}$. We write $\mathfrak{p}_1 \prec \mathfrak{p}_2$ if $e_{1,j_0} < e_{2,j_0}$. We assume that any triple $(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ satisfies $\mathfrak{p}_1 \preceq \mathfrak{p}_2 \preceq \mathfrak{p}_3$.

Let $\mathfrak{p}_i := (e_{i,1}, \ldots, e_{i,n_i})$ be a partition of $d \geq 3$ for each $i = 1, 2, 3$, and put $\mathfrak{P} := (\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$. Let $\mathcal{F}_{\mathfrak{P}} \subset \mathbb{P}_* \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d+3))$ be the family of Artal arrangements of type $\mathfrak{P}$. Here $\mathbb{P}_* \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d+3))$ is the projective space of one-dimensional subspaces of the vector space $\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d+3))$, which parameterizes all plane curves of degree $d+3$. Let s_i be the greatest common divisor $\mathrm{GCD}(e_{i,1}, \ldots, e_{i,n_i})$ for each $i = 1, 2, 3$, and put $s := \mathrm{GCD}(s_1, s_2, s_3)$. Let $\mathcal{A} := B + \mathcal{L}$ be an Artal arrangement of type $\mathfrak{P}$, where $\mathcal{L} := L_1 + L_2 + L_3$. Let $\mathcal{G}_{\mathcal{A}}$ denote the incidence graph $\mathcal{G}_{\phi_{\mathcal{A}}, \mathcal{L}}$ of $\mathcal{L}$ with respect to $\phi_{\mathcal{A}} : X_{\mathcal{A}} \to \mathbb{P}^2$. Let $\gamma_{\mathcal{A}}^+$ be the following cycle on $\mathcal{G}_{\mathcal{A}}$:

$$\gamma_{\mathcal{A}}^+ := (v_{P_1}, v_{L_1}, v_{P_2}, v_{L_2}, v_{P_3}, v_{L_3}, v_{P_1}),$$

where P_1, P_2 and P_3 are the intersections $L_3 \cap L_1$, $L_1 \cap L_2$ and $L_2 \cap L_3$, respectively.

To compute net voltage classes of closed walks on $\mathcal{G}_{\mathcal{A}}$ for $\phi_{\mathcal{A}}$, it is enough to compute the net voltage classes of $\gamma_{\mathcal{A}}^+$ and its inverse walk $\gamma_{\mathcal{A}}^- := (\gamma_{\mathcal{A}}^+)^{-1}$ by Lemma 2.11 and 2.12. Note that the splitting number of L_i for $\phi_{\mathcal{A}}$ is equal to s_i by [18, Theorem 2.7]. By Theorem 3.13, the net voltage class of $\gamma_{\mathcal{A}}^+$ for $\phi_{\mathcal{A}}$ forms into

$$\mathrm{NV}_{\phi_{\mathcal{A}}}(\gamma_{\mathcal{A}}^+) = [\beta] + s\mathbb{Z}_d$$

for some integer β with $0 \leq \beta < s$. By Lemma 2.10, we obtain $\mathrm{NV}_{\phi_{\mathcal{A}}}(\gamma_{\mathcal{A}}^-) = [-\beta] + s\mathbb{Z}_d$. For $0 \leq \alpha \leq \lfloor s/2 \rfloor$, let $\mathcal{F}_{\mathfrak{P}}^{\alpha} \subset \mathcal{F}_{\mathfrak{P}}$ be the set of Artal arrangements $\mathcal{A}$ of type $\mathfrak{P}$ satisfying

$$\{\mathrm{NV}_{\phi_{\mathcal{A}}}(\gamma_{\mathcal{A}}^+), \mathrm{NV}_{\phi_{\mathcal{A}}}(\gamma_{\mathcal{A}}^-)\} = \{[\alpha] + s\mathbb{Z}_d, [-\alpha] + s\mathbb{Z}_d\},$$

where $\lfloor s/2 \rfloor$ is the integer part of $s/2$. The family $\mathcal{F}_{\mathfrak{P}}$ is decomposed into the following disjoint union:

$$\mathcal{F}_{\mathfrak{P}} = \coprod_{\alpha=0}^{\lfloor s/2 \rfloor} \mathcal{F}_{\mathfrak{P}}^{\alpha}.$$

Theorem 4.3. *Let $\mathfrak{p}_i = (e_{i,1}, \ldots, e_{i,n_i})$ be three partition of $d \geq 3$ for $i = 1, 2, 3$, and put $\mathfrak{P} := (\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$, $s_i := \mathrm{GCD}(e_{i,1}, \ldots, e_{i,n_i})$ for $i = 1, 2, 3$ and $s := \mathrm{GCD}(s_1, s_2, s_3)$. Then, two Artal arrangements $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{F}_{\mathfrak{P}}$ have the same embedded topology if and only if the splitting graphs $\mathcal{S}_{\mathcal{A}_1}$ and $\mathcal{S}_{\mathcal{A}_2}$ are equivalent, $\mathcal{S}_{\mathcal{A}_1} \sim \mathcal{S}_{\mathcal{A}_2}$. Moreover, the followings hold:*

(i) *In the case where $\mathfrak{p}_i \neq \mathfrak{p}_j$ for any $i \neq j$, $\mathcal{F}_{\mathfrak{P}}^{\alpha}$ consists of two connected components $\mathcal{F}_{\mathfrak{P}}^{\alpha+}$ and $\mathcal{F}_{\mathfrak{P}}^{\alpha-}$ if $0 < \alpha < s/2$, and $\mathcal{F}_{\mathfrak{P}}^{\alpha}$ is connected otherwise, i.e., either $\alpha = 0$ or $\alpha = s/2$ if s is even.*

(ii) *In the case where $\mathfrak{p}_i = \mathfrak{p}_j$ for some $i \neq j$, $\mathcal{F}_{\mathfrak{P}}^{\alpha}$ is connected for each $0 \leq \alpha \leq \lfloor s/2 \rfloor$.*

(iii) *Let $\mathcal{A}$ be an Artal arrangement of $\mathcal{F}_{\mathfrak{P}}^{\alpha}$ $(0 \leq \alpha \leq \lfloor s/2 \rfloor)$, and let $\bar{h} : \mathbb{P}^2 \to \mathbb{P}^2$ be the base change given by the complex conjugate homomorphism $\mathbb{C} \to \mathbb{C}$ $(z \mapsto \bar{z})$, which is a homeomorphism. Then $\bar{h}(\mathcal{A}) \in \mathcal{F}_{\mathfrak{P}}^{\alpha}$. Moreover, in the case where $\mathfrak{p}_i \neq \mathfrak{p}_j$ for $i \neq j$ and $0 < \alpha < s/2$, $\bar{h}(\mathcal{A}) \in \mathcal{F}_{\mathfrak{P}}^{\alpha-}$ if $\mathcal{A} \in \mathcal{F}_{\mathfrak{P}}^{\alpha+}$.*

(iv) *For two Artal arrangements $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{F}_{\mathfrak{P}}$, there is a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ with $h(\mathcal{A}_1) = \mathcal{A}_2$ if and only if $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{F}_{\mathfrak{P}}^{\alpha}$ for some $0 \leq \alpha \leq \lfloor s/2 \rfloor$.*

In order to prove Theorem 4.3, we prove four lemmas. We first seek a simple defining equation of an Artal arrangement, up to projective transforms of $\mathbb{P}^2$. Let L_x, L_y and L_z be the lines defined by $x = 0$, $y = 0$ and $z = 0$, respectively, and put $\mathcal{L}_{xyz} := L_x + L_y + L_z$.

Lemma 4.4. *Let $\mathcal{A} := B + \mathcal{L}$ be an Artal arrangement of type $\mathfrak{P} = (\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$, where $\mathcal{L} := L_1 + L_2 + L_3$. Put $\mu_{i,j} := e_{i,j}/s_i$. Then, after a certain projective transform of $\mathbb{P}^2$, $\mathcal{L}$ satisfies $L_1 = L_x$, $L_2 = L_y$ and*

$L_3 = L_z$, and B is defined by $F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) = 0$ with

$$F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) := \prod_{j=1}^{n_1}(y + c_{1,j}z)^{e_{1,j}} + \prod_{j=1}^{n_2}(z + c_{2,j}x)^{e_{2,j}} + \prod_{j=1}^{n_3}(x + c_{3,j}y)^{e_{3,j}} - x^d - y^d - z^d + xyzg_0,$$

where β is an integer with $0 \leq \beta < s$, g_0 is a homogeneous polynomial of degree $d-3$ in x, y, z, and $c_{i,j}$ are complex numbers satisfying the following conditions;

$$(5) \quad \prod_{j=1}^{n_i} c_{i,j}^{\mu_{i,j}} = 1 \quad (i = 1, 2), \quad \prod_{j=1}^{n_3} c_{3,j}^{\mu_{3,j}} = \zeta_d^{\beta\mu_3}, \quad c_{i,j} \neq c_{i,j'} \ \text{ if } j \neq j',$$

where $\mu_i := d/s_i = \sum_{j=1}^{n_i} \mu_{i,j}$ for $i = 1, 2, 3$.

Proof. It is clear that $\mathcal{L}$ satisfies $L_1 = L_x$, $L_2 = L_y$ and $L_3 = L_z$ after a projective transform. Let $Q_{i,j}$ be the intersection point of B and L_i with $\mathrm{I}_{Q_{i,j}}(B, L_i) = e_{i,j}$ for each $i = 1, 2, 3$ and $j = 1, \dots, n_i$, and let $a_{i,j}$ be the complex number such that

$$Q_{1,j} = (0{:} -a_{1,j}{:}1), \quad Q_{2,j} = (1{:}0{:} -a_{2,j}), \quad Q_{3,j} = (-a_{3,j}{:}1{:}0).$$

Let $F = 0$ be a defining equation of B. Since $B \cap L_i \cap L_j = \emptyset$ $(i \neq j)$, we may assume that the coefficients of x^d, y^d and z^d in F are equal to 1. Since $\mathrm{I}_{Q_{i,j}}(B, L_i) = e_{i,j}$, a homogenous polynomial F forms into

$$\begin{aligned} F &= \prod_{j=1}^{n_1}(y + a_{1,j}z)^{e_{1,j}} + g_1 + xyzg_0 \\ &= \prod_{j=1}^{n_2}(z + a_{2,j}x)^{e_{2,j}} + g_2 + xyzg_0 \\ &= \prod_{j=1}^{n_3}(x + a_{3,j}y)^{e_{3,j}} + g_3 + xyzg_0, \end{aligned}$$

where g_1, g_2 and g_3 are the sum of terms of F which are not divisible by xyz, but by x, y and z, respectively. Note that $\prod_{j=1}^{n_i} a_{i,j}^{e_{i,j}} = 1$. Then the coefficient of $y^k z^{d-k}$ $(0 < k < d)$ in F is equal to the one in $\prod_{j=1}^{n_1}(y + a_{i,j}z)^{e_{i,j}}$. Similarly, the coefficients of $x^k y^{d-k}$ and $x^k z^{d-k}$ are determined by the above equation. Hence we have

$$F := \prod_{j=1}^{n_1}(y+a_{1,j}z)^{e_{1,j}} + \prod_{j=1}^{n_2}(z+a_{2,j}x)^{e_{2,j}} + \prod_{j=1}^{n_3}(x+a_{3,j}y)^{e_{3,j}} - x^d - y^d - z^d + xyzg_0.$$

Since $\prod_{j=1}^{n_i} a_{i,j}^{e_{i,j}} = (\prod_j^{n_i} a_{i,j}^{\mu_{i,j}})^{s_i} = 1$, we have

$$\prod_{j=1}^{n_i} a_{i,j}^{\mu_{i,j}} = (\zeta_d^{\mu_i})^{\beta_i}$$

for some integer $0 \leq \beta_i < s_i$ $(i = 1, 2, 3)$. After the projective transform defined by $x \mapsto \zeta_d^{-\beta_1-\beta_2}x$, $y \mapsto y$ and $z \mapsto \zeta_d^{-\beta_1}z$, we obtain

$$F = \prod_{j=1}^{n_1}(y+b_{1,j}z)^{e_{1,j}} + \prod_{j=1}^{n_2}(z+b_{2,j}x)^{e_{2,j}} + \prod_{j=1}^{n_3}(x+b_{3,j}y)^{e_{3,j}} - x^d - y^d - z^d + xyzg_0,$$

where $b_{1,j} := \zeta_d^{-\beta_1}a_{1,j}$, $b_{2,j} := \zeta_d^{-\beta_2}a_{2,j}$ and $b_{3,j} := \zeta_d^{\beta_1+\beta_2}a_{3,j}$. We have $\prod_{j=1}^{n_i} b_{i,j}^{\mu_{i,j}} = 1$ for $i = 1, 2$, and $\prod_{j=1}^{n_3} b_{3,j}^{\mu_{3,j}} = (\zeta_d^{\mu_3})^{\beta'}$ for some integer $0 \leq \beta' < s_3$. Suppose that $ks \leq \beta' < (k+1)s$. Let b_1, b_2 and b_3 be three integers so that $b_1s_1 + b_2s_2 + b_3s_3 = ks$. By the projective transform given by $x \mapsto \zeta_d^{b_1s_1+b_2s_2}x$, $y \mapsto y$ and $z \mapsto \zeta_d^{b_1s_1}z$, we obtain $F = F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0)$, where $c_{i,j} := \zeta_{b_is_i}b_{i,j}$ for $i = 1, 2$, $c_{3,j} := \zeta_d^{-b_1s_1-b_2s_2}b_{3,j}$, and $\beta = \beta' - ks$. □

Since $\mathrm{Aut}(\mathbb{P}^2) \cong \mathrm{PGL}(3, \mathbb{C})$ is connected, Lemma 4.4 implies that each connected component of $\mathcal{F}_{\mathfrak{P}}$ contains a member $\mathcal{A}$ defined by $xyzF_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) = 0$ for some integer $0 \leq \beta < s$.

Next, we prove that the curve defined by $F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) = 0$ is smooth for a general polynomial g_0.

Lemma 4.5. *Fix an integer β with $0 \leq \beta < s$, and let $c_{i,j}$ be complex numbers satisfying (5). Then the equation $F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) = 0$ defines a smooth curve B on $\mathbb{P}^2$ for a general homogeneous polynomial g_0.*

Proof. We consider the linear system Λ consisting of curves defined by $aF_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) + bxyzg = 0$ for $(a{:}b) \in \mathbb{P}^1$ and $g \in \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3)) \setminus \{0\}$. Since the base points of Λ are $Q_{1,j} = (0{:}-c_{1,j}{:}1), Q_{2,j} = (1{:}0{:}-c_{2,j}), Q_{3,j} = (-c_{3,j}{:}1{:}0)$, a general member of Λ is smooth except for the base points $Q_{i,j}$ by Bertini's theorem (see [12]). Since $xyzg = 0$ defines a curve smooth at all base points $Q_{i,j}$ if $g(Q_{i,j}) \neq 0$ for any i, j, a general member of Λ is smooth at $Q_{i,j}$. Therefore $F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) = 0$ defines a smooth curve B on $\mathbb{P}^2$ for a general homogeneous polynomial g_0. □

Remark 4.6. If $F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) = 0$ defines a smooth curve B, then $\mathcal{A} = B + \mathcal{L}_{xyz}$ is an Artal arrangement of type $\mathfrak{P} = (\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$. In this case, we put $\gamma_{\mathcal{A}}^+ := (v_{P_1}, v_{L_x}, v_{P_2}, v_{L_y}, v_{P_3}, v_{L_z}, v_{P_1})$.

Lemma 4.7. *Fix an integer β with $0 \leq \beta < s$ and $c_{i,j}$ satisfying (5). Put $\alpha := \beta$ if $\beta \leq \lfloor s/2 \rfloor$, and $\alpha := s - \beta$ if $\beta > \lfloor s/2 \rfloor$. Assume that $F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) = 0$ defines a smooth curve $B \subset \mathbb{P}^2$. Then the equation*

$\mathrm{NV}_{\phi_{\mathcal{A}}}(\gamma_{\mathcal{A}}^{+}) = [\beta] + s\mathbb{Z}_d$ *holds. In particular, the Artal arrangement* $\mathcal{A} := B + \mathcal{L}_{xyz}$ *is a member of* $\mathcal{F}_{\mathfrak{P}}^{\alpha}$.

Proof. Let L be the line defined by $x + y + z = 0$, which intersects transversally with $\mathcal{L}_{xyz}$, and is not a component of B. We compute the net voltage class $\mathrm{NV}_{\phi_{\mathcal{A}}}(\gamma_{\mathcal{A}}^{+})$ by using Theorem 3.13. Let P_1, P_2 and P_3 be the singular points (0:1:0), (0:0:1) and (1:0:0) of $\mathcal{L}_{xyz}$, and put $x' := x/(x + y + z)$, $y' := y/(x + y + z)$ and $z' := z/(x + y + z)$. The system (x', y') is a local coordinate of $U := \mathbb{P}^2 \setminus L$ since $z' = 1 - x' - y'$. The equation $F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0)/(x+y+z)^d = 0$ is a defining equation of B on U. We seek h_i in (2) for L_i $(i = x, y, z)$, and compute $h_i(P_j)$ for $P_j \in L_i$. Since L_x is defined by $x' = 0$ and $s_1 = \mathrm{GCD}(e_{1,1}, \dots, e_{1,n_1})$, we obtain

$$h_x = \prod_{j=1}^{n_1} (y' + c_{1,j} z')^{\mu_{1,j}}.$$

Thus we have $h_x(P_1) = h_x(P_2) = 1$. Similarly, we have $h_y(P_2) = h_y(P_3) = 1$, $h_z(P_3) = 1$ and $h_z(P_1) = (\zeta_d^{\mu_3})^{\beta}$. Hence, by Theorem 3.13, we obtain $\mathrm{NV}_{\phi_{\mathcal{A}}}(\gamma_{\mathcal{A}}^{+}) = [\beta] + s\mathbb{Z}_d$. Moreover, we obtain $\mathrm{NV}_{\phi_{\mathcal{A}}}(\gamma_{\mathcal{A}}^{-}) = [-\beta] + s\mathbb{Z}_d$ by Lemma 2.10. Therefore we have

$$\{\mathrm{NV}_{\phi_{\mathcal{A}}}(\gamma_{\mathcal{A}}^{+}), \mathrm{NV}_{\phi_{\mathcal{A}}}(\gamma_{\mathcal{A}}^{-})\} = \{[\beta]+s\mathbb{Z}, [-\beta]+s\mathbb{Z}_d\} = \{[\alpha]+s\mathbb{Z}_d, [-\alpha]+s\mathbb{Z}_d\}.$$

□

Next we prove that $B + \mathcal{L}_{xyz}$ and $B' + \mathcal{L}_{xyz}$ are two members of a connected component of $\mathcal{F}_{\mathfrak{P}}$ if B and B' are smooth curves defined by $F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) = 0$ and $F_{\mathfrak{P}}(\beta, \{c'_{i,j}\}, g'_0) = 0$, respectively.

Lemma 4.8. *Fix an integer* $0 \leq \beta < s$. *If* B *and* B' *are smooth curves defined by* $F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) = 0$ *and* $F_{\mathfrak{P}}(\beta, \{c'_{i,j}\}, g'_0) = 0$*, respectively, then the Artal arrangements* $\mathcal{A} := B + \mathcal{L}_{xyz}$ *and* $\mathcal{A}' := B' + \mathcal{L}_{xyz}$ *are members of a connected component of* $\mathcal{F}_{\mathfrak{P}}$. *In particular, there is a homeomorphism* $h : (\mathbb{P}^2, \mathcal{A}) \to (\mathbb{P}^2, \mathcal{A}')$ *such that* $h_*([m_B]) = [m_{B'}]$ *for meridians* m_B *and* $m_{B'}$ *of* B *and* B'*, respectively.*

Proof. Let $U_{\mathrm{sm}} \subset (\mathbb{C}^{\times})^{n_1+n_2+n_3} \times \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3))$ be the following subset:

$$U_{\mathrm{sm}} := \left\{ ((c_{i,j}), g_0) \;\middle|\; \begin{array}{l} (c_{i,j}) \in \mathbb{C}^{n_1+n_2+n_3} \text{ satisfies equation (5), and} \\ F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) = 0 \text{ defines a smooth curve} \end{array} \right\}.$$

It is enough to prove that U_{sm} is connected. Let V'_i be the following subset of $(\mathbb{C}^{\times})^{n_i}$ for $i = 1, 2, 3$:

$$V'_i := \left\{ (c_{i,1}, \dots, c_{i,n_i}) \in (\mathbb{C}^{\times})^{n_i} \;\middle|\; \prod_{j=1}^{n_i} c_{i,j}^{\mu_{i,j}} = (\zeta_d^{\mu_i})^{\beta_i} \right\},$$

where $\beta_1 = \beta_2 = 0$ and $\beta_3 = \beta$. Consider the projection $\mathrm{pr}_i : V_i' \to (\mathbb{C}^\times)^{n_i-1}$ defined by $(c_{i,1}, \dots, c_{i,n_i-1}, c_{i,n_i}) \mapsto (c_{i,1}, \dots, c_{i,n_i-1})$. Since $c_{i,n_i}^{\mu_{i,n_i}} = (\zeta_d^{\mu_i})^{\beta_i} \prod_{j=1}^{n_i-1} c_{i,j}^{-\mu_{i,j}}$, the preimage of pr_i at a point of $(\mathbb{C}^\times)^{n_i-1}$ consists of just μ_{i,n_i} points. Let $\rho : \pi_1((\mathbb{C}^\times)^{n_i-1}) \to \mathbb{Z}_{\mu_{i,n_i}}$ be the homomorphism which maps a meridian of $\{c_{i,j} = 0\}$ to $[-\mu_{i,j}] \in \mathbb{Z}_{\mu_{i,n_i}}$. Since $\mathrm{GCD}(\mu_{i,1}, \dots, \mu_{i,n_i}) = 1$, ρ is surjective. Moreover, $\mathrm{pr}_i : V_i' \to (\mathbb{C}^\times)^{n_i-1}$ is the unramified $\mathbb{Z}_{\mu_{i,n_i}}$-cover induced by the surjection ρ. For any $P \in (\mathbb{C}^\times)^{n_i-1}$, transitivity of the action of $\pi_1((\mathbb{C}^\times)^{n_i-1})$ on the fiber $\mathrm{pr}_i^{-1}(P)$ implies that there exists a path connecting any two points in $\mathrm{pr}_i^{-1}(P)$. Hence V_i' is smooth and irreducible. Let U_i' be the following subset of $(\mathbb{C}^\times)^{n_i}$:

$$U_i' := \{(c_{i,1}, \dots, c_{i,n_i}) \in (\mathbb{C}^\times)^{n_i} \mid c_{i,j} \neq c_{i,j'} \ (j \neq j')\}.$$

Since U_i' is a Zariski open subset of $(\mathbb{C}^\times)^{n_i}$, $V_i := U_i' \cap V_i'$ is a Zariski open subset of V_i', hence V_i is connected. For any $(c_{i,j}) \in V_1 \times V_2 \times V_3$, $F_{\mathfrak{P}}(\beta, \{c_{i,j}\}, g_0) = 0$ defines a smooth curve for a general g_0 by Lemma 4.5. Therefore, U_{sm} is a non-empty open subset of $V_1 \times V_2 \times V_3 \times \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3))$, and U_{sm} is connected. $\square$

We prove Theorem 4.3 by using Lemmas 4.4, 4.5, 4.7 and 4.8.

Proof of Theorem 4.3. Let $\mathcal{A}_i = B_i + \mathcal{L}_i$ $(i = 1, 2)$ be two Artal arrangements of type $\mathfrak{P}$. Suppose that the splitting graphs $\mathcal{S}_{\mathcal{A}_1}$ and $\mathcal{S}_{\mathcal{A}_2}$ are equivalent, i.e., there exist a homeomorphism $h' : \mathcal{T}(\mathcal{A}_1) \to \mathcal{T}(\mathcal{A}_2)$ and an automorphism $\tau : \mathbb{Z}_d \to \mathbb{Z}_d$ satisfying Conditions (i), (ii), (iii) in Definition 3.8. By the proof of Lemma 3.6, we have $\tau([1]) = [\pm 1]$ since $\rho_{\mathcal{A}_i}([m_{B_i}]) = [1]$. By Corollary 2.9, we obtain the following equation:

$$\{\mathrm{NV}_{\phi_{\mathcal{A}_1}}(\gamma_{\mathcal{A}_1}^+), \mathrm{NV}_{\phi_{\mathcal{A}_1}}(\gamma_{\mathcal{A}_1}^-)\} = \{\mathrm{NV}_{\phi_{\mathcal{A}_2}}(\gamma_{\mathcal{A}_2}^+), \mathrm{NV}_{\phi_{\mathcal{A}_2}}(\gamma_{\mathcal{A}_2}^-)\}. \tag{6}$$

Hence we obtain $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{F}_{\mathfrak{P}}^\alpha$ for some $0 \leq \alpha \leq \lfloor s/2 \rfloor$. Moreover, the family $\mathcal{F}_{\mathfrak{P}}^\alpha$ is non-empty by Lemmas 4.5 and 4.7.

Note that, if there exists a homeomorphism $h : (\mathbb{P}^2, \mathcal{A}_1) \to (\mathbb{P}^2, \mathcal{A}_2)$ for Artal arrangements $\mathcal{A}_i \in \mathcal{F}_{\mathfrak{P}}$ $(i = 1, 2)$, then $\mathcal{S}_{\mathcal{A}_1} \sim \mathcal{S}_{\mathcal{A}_2}$ by Theorem 3.4 since h must satisfy $h(B_1) = B_2$ and the isomorphism $h_* : \pi_1(\mathbb{P}^2 \setminus B_1) \to \pi_1(\mathbb{P}^2 \setminus B_2)$ is given by $h_*([m_{B_1}]) = [m_{B_2}]^{\pm 1}$ (cf. Remark 3.7). Furthermore, if two Artal arrangements $\mathcal{A}_i := B_i + \mathcal{L}_i \in \mathcal{F}_{\mathfrak{P}}$ is members of the same connected component of $\mathcal{F}_{\mathfrak{P}}$, then there exists a homeomorphism $h : (\mathbb{P}^2, \mathcal{A}_1) \to (\mathbb{P}^2, \mathcal{A}_2)$ with $h_*([m_{B_1}]) = [m_{B_2}]$. Thus it is enough to prove assertions (i), (ii), (iii) and (iv) in Theorem 4.3.

(i) Suppose that $\mathfrak{p}_i \neq \mathfrak{p}_j$ for any $i \neq j$. Let $\mathcal{A}_i := B_i + \sum_{j=1}^3 L_{i,j}$ be an Artal arrangement of type $\mathfrak{P}$ for each $i = 1, 2$. Assume that there is a homeomorphism $h : (\mathbb{P}^2, \mathcal{A}_1) \to (\mathbb{P}^2, \mathcal{A}_2)$ such that $h_*([m_{B_1}]) = [m_{B_2}]$ for meridians m_{B_i} of B_i. In this case, h must satisfy $h(L_{1,j}) = L_{2,j}$

for $j = 1, 2, 3$ (see Remark 4.2), and if an automorphism $\tau : \mathbb{Z}_d \to \mathbb{Z}_d$ satisfies $\tau \circ \rho_{\mathcal{A}_1} = \rho_{\mathcal{A}_2} \circ h_*$, then $\tau([1]) = [1]$. Hence we have $\theta_h(\gamma^+_{\mathcal{A}_1}) = \gamma^+_{\mathcal{A}_2}$ and $\mathrm{NV}_{\phi_{\mathcal{A}_1}}(\gamma^+_{\mathcal{A}_1}) = \mathrm{NV}_{\phi_{\mathcal{A}_2}}(\gamma^+_{\mathcal{A}_2})$.

For $0 < \alpha < s/2$, let $\mathcal{F}^{\alpha\pm}_{\mathfrak{P}}$ be the subsets consisting of $\mathcal{A} \in \mathcal{F}_{\mathfrak{P}}$ with $\mathrm{NV}_{\phi_{\mathcal{A}}}(\gamma^+_{\mathcal{A}}) = [\pm\alpha] + s\mathbb{Z}_d$, respectively. By Lemmas 4.4, 4.5, 4.7 and 4.8, $\mathcal{F}^{\alpha\pm}_{\mathfrak{P}}$ are non-empty and connected. Since $[\alpha] + s\mathbb{Z}_d \neq [-\alpha] + s\mathbb{Z}_d$ for $0 < \alpha < s/2$, $\mathcal{F}^{\alpha+}_{\mathfrak{P}} \cap \mathcal{F}^{\alpha-}_{\mathfrak{P}} = \emptyset$ by Lemma 4.8.

For $\alpha = 0$ or $\alpha = s/2$ if s is even, we have $[\alpha] + s\mathbb{Z}_d = [-\alpha] + s\mathbb{Z}_d$. Hence, by Lemmas 4.4 and 4.7, an Artal arrangement $\mathcal{A} \in \mathcal{F}^{\alpha}_{\mathfrak{P}}$ is projective equivalent to a curve defined by $F_{\mathfrak{P}}(\alpha, \{c_{i,j}\}, g_0) = 0$. Therefore, $\mathcal{F}^{\alpha}_{\mathfrak{P}}$ is connected by Lemma 4.8.

(ii) As in Remark 4.2, we fix the order of partitions $\mathfrak{p}_1 \preceq \mathfrak{p}_2 \preceq \mathfrak{p}_3$. It is enough to check the cases of $\mathfrak{p}_1 = \mathfrak{p}_2$ and $\mathfrak{p}_2 = \mathfrak{p}_3$. Suppose that $\mathfrak{p}_2 = \mathfrak{p}_3$. We have $n_2 = n_3$ and $e_{2,j} = e_{3,j}$ for $j = 1, \dots, n_2$. Let $\mathcal{A}$ be an Artal arrangement defined by $xyzF_{\mathfrak{P}}(\alpha, \{c_{i,j}\}, g_0) = 0$. Let $h : \mathbb{P}^2 \to \mathbb{P}^2$ be the projective transformation defined by $x \mapsto \zeta^{\alpha}_d x$, $y \mapsto z$ and $z \mapsto y$. The image $\mathcal{A}' := h(\mathcal{A})$ is defined by $xyzF_{\mathfrak{P}}(-\alpha, \{c'_{i,j}\}, g'_0) = 0$ satisfying (5), where $c'_{1,j} := c^{-1}_{1,j}$, $c'_{2,j} := \zeta^{\alpha}_d c^{-1}_{3,j}$, $c'_{3,j} := \zeta^{-\alpha}_d c^{-1}_{2,j}$ and g'_0 is the image of g_0. Then the Artal arrangemnts $\mathcal{A}$ and $\mathcal{A}'$ are members of the same connected component of $\mathcal{F}^{\alpha}_{\mathfrak{P}}$ since $\mathrm{PGL}(3, \mathbb{C})$ is connected. Therefore, by Lemmas 4.4, 4.7 and 4.8, $\mathcal{F}^{\alpha}_{\mathfrak{P}}$ is connected. In the case of $\mathfrak{p}_1 = \mathfrak{p}_2$, we can prove the connectivity of $\mathcal{F}^{\alpha}_{\mathfrak{P}}$ by the same argument.

(iii) Let $\mathcal{A} := B + L_1 + L_2 + L_3$ be an Artal arrangement in $\mathcal{F}^{\alpha}_{\mathfrak{P}}$ with $\mathrm{NV}_{\phi_{\mathcal{A}}}(\gamma^+_{\mathcal{A}}) = [\alpha] + s\mathbb{Z}_d$, and put $\bar{B} := \bar{h}(B)$. Let m_B be a meridian of B at a point $P \in B$. By the definition of meridians, $(\bar{h} \circ m_B)^{-1}$ is a meridian of $\bar{B}$ at $\bar{h}(P) \in \bar{B}$. Hence we have $\tau([1]) = [-1]$ for the automorphism $\tau : \mathbb{Z}_d \to \mathbb{Z}_d$ such that $\rho_{\bar{h}(\mathcal{A})} \circ \bar{h}_* = \tau \circ \rho_{\mathcal{A}}$, Thus we obtain $\mathrm{NV}_{\phi_{\bar{h}(\mathcal{A})}}(\gamma^+_{\bar{h}(\mathcal{A})}) = [-\alpha] + s\mathbb{Z}_d$, and $\bar{h}(\mathcal{A}) \in \mathcal{F}^{\alpha}_{\mathfrak{P}}$. In particular, in the case where $\mathfrak{p}_i \neq \mathfrak{p}_j$ for any $i \neq j$ and $0 < \alpha < s/2$, if $\mathcal{A} \in \mathcal{F}^{\alpha+}_{\mathfrak{P}}$, then $\bar{h}(\mathcal{A}) \in \mathcal{F}^{\alpha-}_{\mathfrak{P}}$.

(iv) The assertion follows from (i), (ii) and (iii). $\square$

As a corollary of Theorem 4.3, we obtain Zariski k-plets of Artal arrangements.

Corollary 4.9. *Let $\mathfrak{p}_i$ be a partition $(e_{i,1}, \dots, e_{i,n_i})$ of $d \geq 3$ for each $i = 1, 2, 3$. Let s be the greatest common divisor of $e_{i,j}$, $i = 1, 2, 3$ and $j = 1, \dots, n_i$. Then there is a Zariski $(\lfloor s/2 \rfloor + 1)$-plet $(\mathcal{A}_0, \dots, \mathcal{A}_{\lfloor s/2 \rfloor})$ of Artal arrangements $\mathcal{A}_i$ of type $\mathfrak{P}$, i.e., $(\mathcal{A}_i, \mathcal{A}_j)$ is a Zariski pair for any $0 \leq i < j \leq \lfloor s/2 \rfloor$.*

Proof. Let $\mathcal{A}_\alpha$ be a member of $\mathcal{F}^{\alpha}_{\mathfrak{P}}$ for each $\alpha = 0, \dots, \lfloor s/2 \rfloor$. Then $(\mathcal{A}_0, \dots, \mathcal{A}_{\lfloor s/2 \rfloor})$ is a Zariski $(\lfloor s/2 \rfloor + 1)$-plet by Theorem 4.3. $\square$

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