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Axiomatic Method of Measure and Integration (VII). Definition and Existence Theorem of LS-measure

(Y. Ito, "RS-integral and LS-integral", Chap.8−9)

By

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Abstract

In this paper, we define the LS-measure on $\boldsymbol{R}^d$, $(d \geq 1)$ by prescribing the complete system of axioms. Then we prove the existence theorem of the LS-measure and we determine all the LS-measures. They are the new results.

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Introduction

This paper is the part VII of the series of papers on the axiomatic method of measure and integration on the Euclidean space. As for the details, we refer to Ito [6], [14]. Further we refer to Ito [1] $\sim$ [5], [7] $\sim$ [13] and [15] $\sim$ [23].

In this paper, we define the Lebesgue-Stieltjes measure on the d-dimensional Euclidean space and prove its existence theorem. Further we study the fundamental properties of the d-dimensional Lebesgue-Stieltjes measurable sets. Here we assume $d \geq 1$.

For simplicity, we say that a d-dimensional Lebesgue-Stieltjes measure on the d-dimensional Euclidean space is a d-dimensional LS-measure. Further, for simplicity, we say that a d-dimensional Lebesgue-Stieltjes measurable set is a d-dimensional LS-measurable set.

In this paper, in the sequel, we happen to omit the adjective "d-dimensional".

A LS-measure is a completely additive real-valued measure and every completely additive real-valued measure on the Euclidean space is a certain LS-measure.

Thereby the set of all completely additive real-valued measures on the Euclidean space is determined. Namely this set is the set of all LS-measures on the Euclidean space.

The most fundamental property of the considered measure is the additivity.

The Lebesgue measure is an additive set function defined on the σ-ring of all Lebesgue measurable sets and it is the measure characterized by the conditions such as the positivity, the complete additivity and the invariance with respect to the congruent transformation.

Therefore the value of the Lebesgue measure is determined so that the unit measure is the measure of the unit figure.

The LS-measure is an additive set function defined on the σ-ring of all LS-measurable sets. It is the measure characterized by the two conditions such as the real-valuedness and the complete additivity.

Further the Lebesgue measure is known to be the special example of the LS-measure.

We can construct a LS-measure independent to the dimension of the Euclidean space. Namely, except for the meaning of the symbol, we can obtain its expression independent to the dimension in the framework of the formal calculation of the symbols. When we consider the meaning of these symbols concretely, we have the substantial meanings as for the differences of 1-dimension or 2-dimension etc.

In order to understand the meaning of the calculation really in mathematics, it is really most important to understand the meaning of the symbols. Therefore, the differences of 1-dimension and 2-dimension and so on have the important meanings in the measure theory. The mathematics is not only the problem of the formal calculation of the symbols.

Here I express my heartfelt gratitude to my wife Mutuko for her help of typesetting this manuscript.

1 The definition of a d-dimensional LS-measure and its existence theorem

In this section, we define the concept of a d-dimensional LS-measure and prove its existence theorem. Here we assume $d \geq 1$.

1.1 Intervals, blocks of intervals and Borel sets

In this paragraph, we prepare the terminology necessary for giving the system of axioms of a d-dimensional LS-measure.

At first we study the intervals and the blocks of intervals which are the fundamental subsets of the d-dimensional Euclidean space $\boldsymbol{R}^d$.

We say that a subset E of $\boldsymbol{R}^d$ is an **interval** if E is a direct product set of the form

$$E = \prod_{p=1}^{d} I_p.$$

Here $I_1,\ I_2,\ \cdots,\ I_d$ are intervals of $\boldsymbol{R}$ and each one of them is the subset of $\boldsymbol{R}$ of the form such as

$$\begin{aligned} &(a,\ b) = \{x;\ a < x < b\},\ [a,\ b) = \{x;\ a \leq x < b\}, \\ &(a,\ b] = \{x;\ a < x \leq b\},\ [a,\ b] = \{x;\ a \leq x \leq b\}. \end{aligned} \tag{1.1}$$

Here a and b are some real numbers or $-\infty$ or ∞. Then $-\infty$ or ∞ is not a point of the interval I_p, $(1 \leq p \leq d)$.

Here we denote the interior of E as the formula

$$E^\circ = \prod_{p=1}^{d} I_p^\circ. \tag{1.2}$$

Here $I_p^\circ = (a,\ b)$, $(1 \leq p \leq d)$ and the empty set ϕ is considered to be the interval.

We say that a subset E of $\boldsymbol{R}^d$ is a **block of intervals** if there are mutually disjoint finite intervals $I_1,\ I_2,\ \cdots,\ I_n$ such that E is expressed as the direct sum

$$E = \bigcup_{p=1}^{n} I_p = \sum_{p=1}^{n} I_p = I_1 + I_2 + \cdots + I_n. \tag{1.3}$$

We say that the formula (1.3) is the **division** of E by using the intervals $I_1,\ I_2,\ \cdots,\ I_n$. In general, there are the infinitely many varieties of the divisions of one block of intervals. Here we denote the family of all blocks of intervals in $\boldsymbol{R}^d$ as $\mathcal{R}$. Then $\mathcal{R}$ is the smallest ring including all intervals in $\boldsymbol{R}^d$. Namely $\mathcal{R}$ is the ring of sets generated by the family of sets $\mathcal{P}$. Here $\mathcal{P}$ denotes the family of all intervals in $\boldsymbol{R}^d$.

Then we have the theorem in the following.

Theorem 1.1 *If $\mathcal{R}$ is the family of all blocks of intervals in $\boldsymbol{R}^d$, we have the statements* (1) $\sim$ (3) *in the following*:

(1) $\phi \in \mathcal{R}$.

(2) *If we have $A \in \mathcal{R}$, we have*

$$A^c = \{x \in \boldsymbol{R}^d;\ x \notin A\} \in \mathcal{R}.$$

(3) *If we have $A,\ B \in \mathcal{R}$, we have $A \cup B \in \mathcal{R}$.*

Therefore, by virtue of Theorem 1.1, the family of sets $\mathcal{R}$ is known to be a **ring** of sets.

Corollary 1.1 *Assume that $\mathcal{R}$ is the same as in* Theorem 1.1. *Then we have the statements* (1) $\sim$ (3) *in the following*:

(1) $\boldsymbol{R}^d \in \mathcal{R}$.

(2) *If we have $A,\ B \in \mathcal{R}$, we have $A - B \in \mathcal{R}$. Here the difference $A - B = A \backslash B$ of the sets A and B is defined by the formula*

$$A \backslash B = A \cap B^c = \{x \in \boldsymbol{R}^d;\ x \in A,\ x \notin B\}.$$

(3) *If we have $A_p \in R$, $(p = 1,\ 2,\ \cdots,\ n)$, we have the following*:

$$(1)\ \bigcup_{p=1}^{n} A_p \in \mathcal{R}, \quad (2)\ \bigcap_{p=1}^{n} A_p \in \mathcal{R}.$$

Therefore, because $\mathcal{R}$ satisfies the statement (1) in Corollary 1.1, the ring of sets $\mathcal{R}$ is known to be an **algebra** of sets.

Here we give the definition in the following.

Definition 1.1 We say that a family $\boldsymbol{B}$ of subsets of $\boldsymbol{R}^d$ is a $\boldsymbol{\sigma}$**-ring** if we have the conditions (i) and (ii) in the following:

(i) If we have $A,\ B \in \boldsymbol{B}$, we have $A - B \in \boldsymbol{B}$.

(ii) If we have $A_p \in \boldsymbol{B}$, $(p = 1,\ 2,\ \cdots)$, we have

$$\bigcup_{p=1}^{\infty} A_p \in \boldsymbol{B}.$$

Corollary 1.2 *Assume that $\boldsymbol{B}$ is a σ-ring of subsets of $\boldsymbol{R}^d$. Then, for $A_p \in \boldsymbol{B}$, $(p = 1,\ 2,\ 3,\ \cdots)$, we also have the statements* (1) $\sim$ (3) *in the following*:

(1) $\displaystyle\bigcap_{p=1}^{\infty} A_p \in \boldsymbol{B}$. (2) $\displaystyle\varlimsup_{p\to\infty} A_p \in \boldsymbol{B}$. (3) $\displaystyle\varliminf_{p\to\infty} A_p \in \boldsymbol{B}$.

Further, if $\displaystyle\lim_{p\to\infty} A_p$ *exists, we have*

$$\lim_{p\to\infty} A_p \in \boldsymbol{B}.$$

In Corollary 1.2, we define the **superior limit** $\varlimsup A_p$ and the **inferior limit** $\varliminf A_p$ of a sequence of subsets $\{A_p\}$ by the relations

$$\varlimsup A_p = \varlimsup_{p\to\infty} A_p = \bigcap_{n=1}^{\infty} \bigcup_{p=n}^{\infty} A_p,$$
$$\varliminf A_p = \varliminf_{p\to\infty} A_p = \bigcup_{n=1}^{\infty} \bigcap_{p=n}^{\infty} A_p.$$

Especially if we have the condition

$$\varlimsup_{p\to\infty} A_p = \varliminf_{p\to\infty} A_p,$$

we put

$$\lim_{p\to\infty} A_p = \varlimsup_{p\to\infty} A_p = \varliminf_{p\to\infty} A_p.$$

We call it the **limit** of the sequence of subsets $\{A_p\}$.

Then we denote the smallest σ-ring including a family $\mathcal{F}$ of subsets of $\boldsymbol{R}^d$ as $\sigma(\mathcal{F})$ and we call it as the **$\boldsymbol{\sigma}$-ring generated by the family $\mathcal{F}$ of sets**.

Now we assume that the family $\boldsymbol{B}$ of sets is a σ-ring generated by $\mathcal{P}$ or $\mathcal{R}$. Then we say that an element of $\boldsymbol{B}$ is a **Borel set**.

Corollary 1.3 *Assume that $\boldsymbol{B}$ is the family of all* Borel *sets in $\boldsymbol{R}^d$. Then we have the statements* (1) $\sim$ (4) *in the following*:

(1) $\mathcal{P} \subset \mathcal{R} \subset \boldsymbol{B}$.

(2) $\boldsymbol{B} = \sigma(\mathcal{P}) = \sigma(\mathcal{R})$.

(3) $\boldsymbol{R}^d \in \boldsymbol{B}$.

(4) *If we have $A \in \boldsymbol{B}$, we have $A^c \in \boldsymbol{B}$.*

Therefore $\boldsymbol{B}$ is known to be a **$\boldsymbol{\sigma}$-algebra**. Then we say that $\boldsymbol{B}$ is the **Borel algebra**.

Proposition 1.1 *An arbitrary element of the* Borel *algebra* $\boldsymbol{B}$ *is included in a union of a countable number of certain elements of* $\mathcal{P}$.

Further an arbitrary element of $\boldsymbol{B}$ *is included in a union of a countable number of certain elements of* $\mathcal{R}$.

1.2 Definition of LS-measure

In this paragraph, we define a LS-measure space and a LS-measure. These are given by Definition 1.2 in the following.

Here we prepare the terminology which is used in the definition of a LS-measure.

In general, we consider a certain σ-finite completely additive measure space $(X,\ \mathcal{F},\ \mu)$. Assume that the range of μ is a subset of $\overline{\boldsymbol{R}} = [-\infty,\ \infty]$. Here we assume that the range of μ does not contain ∞ and $-\infty$ simultaneously.

Then, for $A \in \mathcal{F}$, we put

$$|\mu|(A) = \sup \sum_{j=1}^{n} |\mu(A_j)|.$$

Here sup is considered for all the choices of finite divisions of A such as

$$A = A_1 + A_2 + \cdots + A_n,\ \{A_j \in \mathcal{F},\ (1 \le j \le n)\}.$$

Then we say that a set function $|\mu|$ is the **total variation** of μ.

Further, for $A \in \mathcal{F}$, we put

$$\mu^+(A) = \sup_{E \subset A} \mu(E) \ge \mu(\phi) = 0,$$

$$\mu^-(A) = -\inf_{E \subset A} \mu(E) \ge -\mu(\phi) = 0.$$

Here sup and inf are considered for all sets $E \in \mathcal{F}$ such as we have $E \subset A$.

Then we say that two set functions μ^+ and μ^- are **the positive variation** and the **negative variation** of μ respectively.

As for these, we have the theorem in the following.

Theorem 1.2 *We use the notation in the above. Then,* $|\mu|$, μ^+ *and* μ^- *are the completely additive positive measures on* $\mathcal{F}$ *and, for* $A \in \mathcal{F}$, *we have the equalities*

$$\mu(A) = \mu^+(A) - \mu^-(A),$$

$$|\mu|(A) = \mu^+(A) + \mu^-(A).$$

Definition 1.2 (LS-measure) If a family $\mathcal{M}$ of sets on the d-dimensional Euclidean space $\boldsymbol{R}^d$ and a set function μ satisfy the system of axioms (I) $\sim$ (IV) in the following, we say that the triplet $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ is a **d-dimensional LS-measure space**.

Then we say that an element of $\mathcal{M}$ is a **LS-measurable set** and μ is the **d-dimensional LS-measure**.

Here we assume that ν is the total variation of μ and we assume that μ^+ and μ^- are the positive variation and the negative variation of μ respectively.

(I) We have $\boldsymbol{B} \subset \mathcal{M}$. Here the family of sets $\boldsymbol{B}$ is the family of all Borel sets.

(II) We have the axioms (i) and (ii) in the following:

(i) We have either one of the conditions (a) or (b) in the following:

(a) For an arbitrary set $A \in \mathcal{M}$, we have $\infty < \mu(A) \leq \infty$.

(b) For an arbitrary set $A \in \mathcal{M}$, we have $-\infty \leq \mu(A) < \infty$.

(ii) If a countable number of elements $A_1,\ A_2,\ \cdots,\ A_n,\ \cdots$ of $\mathcal{M}$ are mutually disjoint, the direct sum

$$A = \bigcup_{p=1}^{\infty} A_p = \sum_{p=1}^{\infty} A_p$$

is also an element of $\mathcal{M}$ and we have the equality

$$\mu(A) = \sum_{p=1}^{\infty} \mu(A_p).$$

(III) We have $A \in \mathcal{M}$ if and only if, for an arbitrary bounded set $E \in \boldsymbol{B}$, we have the equality

$$\nu^*(A \cap E) = \nu_*(A \cap E).$$

Here ν^* and ν_* denote the outer measure and the inner measure defined by using the measure ν on $\boldsymbol{B}$ respectively. Then the measure ν on $\boldsymbol{B}$ is the restriction of the total variation ν of μ to $\boldsymbol{B}$.

Namely $\nu^*(A \cap E)$ and $\nu_*(A \cap E)$ are defined by the formulas

$$\nu^*(A \cap E) = \inf\{\nu(B);\ B \supset A \cap E,\ B \in \boldsymbol{B}\},$$

$$\nu_*(A \cap E) = \sup\{\nu(B);\ A \cap E \supset B,\ B \in \boldsymbol{B}\}.$$

(IV) For $A \in \mathcal{M}$, we have the equalities

$$\mu(A) = \mu^+(A) - \mu^-(A),$$

$$\nu(A) = \mu^+(A) + \mu^-(A).$$

Especially, we say that a LS-measure is a **positive LS-measure** if its range is included in $[0, \infty]$. Then, ν, μ^+ and μ^- are the positive LS-measures.

For simplicity, we say that a d-dimensional LS-measure space and a d-dimensional LS-measure are a **LS-measure space** and a **LS-measure** respectively. Further we say that a LS-measurable set is a **measurable set**.

We assume that the series

$$\mu(A) = \sum_{p=1}^{\infty} \mu(A_p)$$

converges absolutely or diverges to $\pm\infty$ in the axiom (II), (ii).

Each LS-measure only satisfies either one of the conditions (a) or (b) in the axiom (II), (i) in Definition 1.2.

This condition means that the range of the LS-measure does not contain ∞ and $-\infty$ simultaneously.

Therefore at least one of μ^+ or μ^- has the finite total measure. Then we say that the measure of the finite total measure is a **finite measure**. The condition (ii) of this axiom (II) means that a d-dimensional LS-measure is a completely additive measure.

Corollary 1.4 *We use the notation in* Definition 1.2. *Then, for $A \in \mathcal{M}$, we have the equality*

$$\nu(A) = \nu^*(A) = \sup\{\nu^*(A \cap E);\ E \in \boldsymbol{B} \text{ is bounded}\}$$

$$= \nu_*(A) = \sup\{\nu_*(A \cap E);\ E \in \boldsymbol{B} \text{ is bounded}\}.$$

Corollary 1.5 *Assume that $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ is a LS-measure space. Then we have the statements* (1) $\sim$ (3) *in the following*:

(1) *If $A_1, \cdots, A_n \in \mathcal{M}$ are mutually disjoint, we have the equality*

$$\mu(\sum_{p=1}^{n} A_p) = \sum_{p=1}^{n} \mu(A_p).$$

(*Finite additivity*).

(2) *Assume that λ is either one of ν, μ^+ and μ^-. Then, if we have $A,\ B \in \mathcal{M}$ and $A \supset B$, we have the inequality $\lambda(A) \geq \lambda(B)$. Especially, if we have $\lambda(B) < \infty$, we have the equality $\lambda(A \backslash B) = \lambda(A) - \lambda(B)$.*

(3) *Assume that λ is the same as in* (2). *Then, if we have $A_p \in \mathcal{M}$, $(p \geq 1)$, we have the inequality*

$$\lambda(\bigcup_{p=1}^{\infty} A_p) \leq \sum_{p=1}^{\infty} \lambda(A_p).$$

(*Completely sub-additivity*).

By virtue of (1) in Corollary 1.5, we know that this completely additive measure μ is a finite additive measure.

1.3 Existence theorem of a LS-measure

In this paragraph, we prove the existence theorem of a d-dimensional LS-measure defined in Definition 1.2.

For that purpose, we have only to determine the family $\mathcal{M}$ of the LS-measurable sets and the LS-measure μ in $\boldsymbol{R}^d$ concretely.

At first, we assume that there exists a LS-measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ satisfying the system of axioms in Definition 1.2. Then we must determine the family $\mathcal{M}$ of sets in $\boldsymbol{R}^d$ and we must determine the value of $\mu(A)$ for an arbitrary element A in $\mathcal{M}$.

Then, by virtue of the axiom (III) of Definition 1.2, we have the completely additive measure space $(\boldsymbol{R}^d,\ \boldsymbol{B},\ \mu)$ if we restrict the LS-measure μ to the Borel algebra $\boldsymbol{B}$. We say that this measure space is the **Borel-Stieltjes measure space**. By expressing this in the abbreviated form, we say that this is the **BS-measure space** and the **Borel-Stieltjes measure** μ is the **BS-measure**.

1.3.1 Definition of a BS-measure. Here we give the definition of the BS-measure space in the following.

Definition 1.3 (BS-measure) We define that a completely additive measure space $(\boldsymbol{R}^d,\ \boldsymbol{B},\ \mu)$ is a **BS-measure space** and μ is a **BS-measure** if we have the axioms (i) $\sim$ (iii) in the following:

(i) We have only one of the conditions (a) or (b) in the following:

  (a) For an arbitrary set $A \in \boldsymbol{B}$, we have $-\infty \leq \mu(A) < \infty$.

  (b) For an arbitrary set $A \in \boldsymbol{B}$, we have $-\infty < \mu(A) \leq \infty$.

(ii) If a countable number of elements $A_1,\ A_2,\ \cdots,\ A_n,\ \cdots$ of $\boldsymbol{B}$ are mutually disjoint, the direct sum

$$A = \bigcup_{p=1}^{\infty} A_p = \sum_{p=1}^{\infty} A_p$$

is an element of $\boldsymbol{B}$ and we have the equality

$$\mu(A) = \sum_{p=1}^{\infty} \mu(A_p).$$

(iii) If ν is the total variation of μ and if μ^+ and μ^- are the positive variation of μ and the negative variation of μ respectively, then, for $A \in \boldsymbol{B}$, we have the equalities in the following

$$\mu(A) = \mu^+(A) - \mu^-(A),$$

$$\nu(A) = \mu^+(A) + \mu^-(A).$$

Especially, when the range of a BS-measure μ is a subset of $[0, \infty]$, we say that μ is a positive BS-measure.

Then ν, μ^+ and μ^- in Theorem 1.3 are the positive BS-measures.

Corollary 1.6 *Assume that* $(\boldsymbol{R}^d, \boldsymbol{B}, \mu)$ *is a BS-measure space. Then we have the statements* (1) $\sim$ (3) *in the following*:

(1) *If* $A_1, \cdots, A_n \in \boldsymbol{B}$ *are mutually disjoint, we have the equality*

$$\mu(\sum_{p=1}^{n} A_p) = \sum_{p=1}^{n} \mu(A_p).$$

(*Finite additivity*).

(2) *Assume that* λ *is one of* ν, μ^+ *and* μ^-. *Then, if we have* $A, B \in \boldsymbol{B}$ and $A \supset B$, *we have* $\lambda(A) \geq \lambda(B)$. *Especially, if we have* $\nu(B) < \infty$, *we have* $\lambda(A \backslash B) = \lambda(A) - \lambda(B)$.

(3) *Assume that* λ *is the same as in* (2). *Then, for* $A_p \in \boldsymbol{B}$, $(p \geq 1)$, *we have the inequality*

$$\lambda(\bigcup_{p=1}^{\infty} A_p) \leq \sum_{p=1}^{\infty} \lambda(A_p).$$

(*Complete sub-additivity*)

Especially, for $A_1, A_2, \cdots, A_n \in \boldsymbol{B}$, *we have the inequality*

$$\lambda(\bigcup_{p=1}^{n} A_p) \leq \sum_{p=1}^{n} \lambda(A_p).$$

(*Finite sub-additivity*)

1.3.2 Existence theorem of a BS-measure. In this paragraph, we prove the existence theorem of a BS-measure.

By virtue of the definition of a BS-measure, we have only to prove the existence theorem of a positive BS-measure.

When $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ is a positive LS-measure space, we have the positive BS-measure space $(\boldsymbol{R}^d,\ \boldsymbol{B},\ \mu)$ by restricting μ to $\boldsymbol{B}$.

Then the BS-measure μ is characterized by the Theorem 1.3 in the following.

Theorem 1.3 *Assume that μ is a positive* LS-*measure. Then, for an arbitrary set $A \in \boldsymbol{B}$, we have the equality*

$$\mu(A) = \inf \sum_{p=1}^{\infty} \mu(E_p).$$

Here inf *is taken for all countable sequences $\{E_p\}$ of elements in $\mathcal{R}$ whose union contains A.*

1.3.3 Proof of the existence theorem of a LS-measure. By the preparation in the above, we prove the existence theorem of a LS-measure.

Then, by virtue of the axiom (IV) of Definition 1.2, we have only to determine two positive LS-measures μ^+ and μ^- on $\mathcal{M}$.

Therefore, in the sequel, we prove the existence theorem of a positive LS-measure satisfying the system of axioms in Definition 1.2.

For that purpose, we prepare the notation.

We consider a function $f(x) = f(x_1,\ x_2,\ \cdots,\ x_d)$ defined on $\boldsymbol{R}^d$.

Here, for an interval

$$E = \prod_{p=1}^{d} I_p,\ I_p = [a_p,\ b_p),\ (p = 1,\ 2,\ \cdots,\ d)$$

of $\boldsymbol{R}^d$, we put

$$\Delta_{I_p} f(x) = f(x_1,\ \cdots,\ x_{p-1},\ b_p,\ x_{p+1},\ \cdots,\ x_d)$$

$$-f(x_1,\ \cdots,\ x_{p-1},\ a_p,\ x_{p+1},\ \cdots,\ x_d),$$

$$\Delta_E f(x) = \Delta_{I_1}\Delta_{I_2}\cdots\Delta_{I_d} f(x) = \Delta_{I_1}(\Delta_{I_2}\cdots(\Delta_{I_p} f(x))).$$

Especially, when we have $I_p = \{a_p\}$, we put

$$\Delta_{I_p} f(x) = f(x_1,\ \cdots,\ x_{p-1},\ a_p + 0,\ x_{p+1},\ \cdots,\ x_d)$$

$$-f(x_1,\ \cdots,\ x_{p-1},\ a_p,\ x_{p+1},\ \cdots,\ x_d).$$

Here we happen to denote $I_p = \{a_p\} = [a_p,\ a_{p+0})$.

Further, when either one of $a_p,\ b_p,\ (1 \leq p \leq d)$ is equal to ∞ or$-\infty$, we consider the limit such as $a_p \to -\infty$ or $b_p \to \infty$ in the symbols in the above.

Here we prove the two lemmas in the following.

Lemma 1.1 *We use the notation in the following. Assume that a real-valued function* $f(x) = f(x_1,\ x_2,\ \cdots,\ x_d)$ *defined on* $\boldsymbol{R}^d$ *satisfies the conditions* (i) *and* (ii) *in the following*:

(i) $f(x)$ *is a variable-wise left continuous function.*

(ii) *For an arbitrary interval*

$$E = \prod_{p=1}^{d} I_p, \tag{1.4}$$

the condition

$$\Delta_E f(x) \geq 0$$

is satisfied. Here we denote

$$I_p = [x_p,\ y_p)\ or\ I_p = \{x_p\} = [x_p,\ x_p + 0),$$

$$(x_p,\ y_p \in \boldsymbol{R},\ x_p < y_p,\ (1 \leq p \leq d)).$$

We assume that $\mathcal{P}$ *is the family of all intervals in* $\boldsymbol{R}^d$. *Then there exists one and only one conditionally completely additive positive measure* μ *on* $\mathcal{P}$ *such that we have the condition* (1) *in the following*:

(1) *For an interval* E *in the formula* (1.4), *we have the formula*

$$\mu(E) = \Delta_E f(x).$$

Lemma 1.2 *We use the notation in* Lemma 1.1. *Assume that* $\mathcal{R}$ *is the ring of all blocks of intervals in* $\boldsymbol{R}^d$.

Then there exists one and only one conditionally completely additive positive measure μ *on* $\mathcal{R}$ *such that we have the conditions* (1) *and* (2) *in the following*:

(1) *For an interval* E *in the formula* (1.4), *we have the equality*

$$\mu(E) = \Delta_E f(x).$$

(2) *If* $E \in \mathcal{R}$ *has a division by using the finite number of mutually disjoint intervals* $E_1,\ \cdots,\ E_n$ *as follows*:

$$E = E_1 + E_2 + \cdots + E_n,$$

we have the equality

$$\mu(E) = \mu(E_1) + \mu(E_2) + \cdots + \mu(E_n).$$

Further the value of $\mu(E)$ *is determined uniquely and independently with the choice of the divisions of* E *by using the intervals. Then, for* $E \in \mathcal{P}$, $\mu(E)$ *coincides with the value of the interval function defined in* Lemma 1.1.

Theorem 1.4(Existence theorem of RS-measure) *Assume that a function* $f(x)$ *is the same as in* Lemma 1.1. *Then there exists one and only one positive* RS-*measure space* $(\boldsymbol{R}^d,\ \mathcal{M}_0,\ \mu)$ *such that we have the condition* (1) *in the following*:

(1) *For an interval* E *in the formula* (1.4), *we have the equality*

$$\mu(E) = \Delta_E f(x).$$

Further the inverse is also true.

Here we extend the conditionally completely additive positive RS-measure on $\mathcal{R}$ defined in Lemma 1.2 to the positive BS-measure μ on the family of all Borel sets $\boldsymbol{B}$.

Lemma 1.3 *We assume that a positive* RS-*measure on* $\mathcal{R}$ is μ. *Then, for an arbitrary set* $A \in \boldsymbol{B}$, *we put*

$$\widetilde{\mu}(A) = \inf \sum_{p=1}^{\infty} \mu(E_p).$$

Here inf *is taken for all sequences* $\{E_p\}$ *of countable elements in* $\mathcal{R}$ *whose unions contain* A. *Then* $\widetilde{\mu}$ *is a positive* BS-*measure on* $\boldsymbol{B}$.

For the simplicity of the expression, we denote $\widetilde{\mu}$ in Lemma 1.3 as μ.

In Lemma 1.3, we prove the existence of the positive BS-measure space $(\boldsymbol{R}^d,\ \boldsymbol{B},\ \mu)$.

After the preparation in the above, we prove the existence theorem of the positive LS-measure. Namely, by completing the positive BS-measure space $(\boldsymbol{R}^d,\ \boldsymbol{B},\ \mu)$ given in the above, we prove the existence theorem of the positive LS-measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$.

Theorem 1.5 (Existence theorem of a LS-measure) *Assume that a function* $f(x)$ *is the same as in* Lemma 1.1. *Then there exists one and only one positive* LS-*measure space* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ *such that we have the condition* (1) *in the following*:

(1) *For an interval* E *in the formula* (1.4), *we have the equality*

$$\mu(E) = \Delta_E f(x).$$

Further the inverse is also true.

The positive LS-measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ in Theorem 1.5 is given in the following.

If A is a subset of $\boldsymbol{R}^d$, we put

$$\mu^*(A) = \inf\{\mu(B);\ B \supset A,\ B \in \boldsymbol{B}\},$$

$$\mu_*(A) = \sup\{\mu(B);\ A \supset B,\ B \in \boldsymbol{B}\}.$$

We say that they are the **outer measure** and the **inner measure** of A respectively.

Then we say that an arbitrary subset A of $\boldsymbol{R}^d$ is a LS-measurable set if and only if, for an arbitrary bounded set $E \in \boldsymbol{B}$, we have the equality

$$\mu^*(A \cap E) = \mu_*(A \cap E).$$

Further we say that, for $A \in \mathcal{M}$,

$$\mu(A) = \sup\{\mu^*(A \cap E);\ E \in \boldsymbol{B} \text{ is bounded}\}$$

is the positive LS-measure of A.

Then the triplet $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ is the positive LS-measure space in Theorem 1.5.

We say that the function $f(x)$ considered in Theorem 1.5 is the **distribution function** of the positive LS-measure μ.

Corollary 1.7 *Assume that A is a subset of $\boldsymbol{R}^d$. Then, for the outer measure and the inner measure of A defined in the proof of* Theorem 1.5, *we have the equalities* (1) *and* (2) *in the following*:

$$(1)\quad \mu^*(A) = \sup\{\mu^*(A \cap E);\ E \in \boldsymbol{B} \textit{ is bounded}\}.$$

$$(2)\quad \mu_*(A) = \sup\{\mu_*(A \cap E);\ E \in \boldsymbol{B} \textit{ is bounded}\}.$$

Corollary 1.8 *We use the notation in* Corollary 1.7. *Then, if a subset A of $\boldsymbol{R}^d$ is a* LS-*measurable set, we have the equality*

$$\mu(A) = \mu^*(A) = \mu_*(A)$$

for the positive LS-*measure μ.*

Theorem 1.6 *Assume that μ is a positive* LS-*measure on $\boldsymbol{R}^d$ and a function $f(x)$ is the distribution function of μ. Then $f(x)$ is continuous if and only if, for each point $x_j \in \boldsymbol{R}$, we have $\mu(\{x_j\} \times E_j) = 0,\ (1 \le j \le d)$.*

Here we put $x' = (x_1,\ \cdots,\ x_{j-1},\ x_{j+1}, \cdots, x_d)$ and $E_{x'}$ is an interval in $\boldsymbol{R}^{d-1}_{x'}$.

In the same way as in Theorem 1.5, we can prove the following theorems.

Theorem 1.7 (Existence theorem of a LS-measure) *Assume that a real-valued function $f(x)$ defined on $\boldsymbol{R}^d$ satisfies the conditions* (i) *and* (ii) *in the following*:

(i) *$f(x)$ is a variable-wise right continuous function.*

(ii) *For an arbitrary interval*

$$E = \prod_{p=1}^{d} I_p, \tag{1.5}$$

the condition

$$\Delta_E f(x) \geq 0$$

is satisfied. Here we put

$$I_p = (x_p,\ y_p] \text{ or } I_p = \{x_p\} = (x_p - 0,\ x_p],$$

$$(x_p,\ y_p \in \boldsymbol{R} \text{ and } x_p < y_p,\ (1 \leq p \leq d)\}.$$

Then there exists one and only one positive LS-*measure space* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ *such that we have the condition* (1) *in the following*:

(1) *For an interval E in the formula* (1.5), *we have the equality*

$$\mu(E) = \Delta_E f(x).$$

Further the inverse is also true.

For $f(x)$ and μ which satisfy the conditions of Theorem 1.7, we have the similar result as Theorem 1.6.

Theorem 1.8(Existence theorem of a LS-measure) *Assume that a function $f(x)$ defined on $\boldsymbol{R}^d$ satisfies the conditions* (i) *and* (ii) *in the following* :

(i) *$f(x)$ is a variable-wise left continuous function of locally bounded variation.*

(ii) *The two functions $f^+(x)$ and $f^-(x)$ are the positive variation and the negative variation of $f(x)$ respectively. We have the equality*

$$f(x) = f^+(x) - f^-(x).$$

Then there exists one and only one LS-*measure space* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ *such that we have the condition* (1) *in the following*:

(1) *For an interval E in the formula* (1.4), *we have the equality*

$$\mu(E) = \Delta_E f(x).$$

The inverse is also true.

When we consider the positive variation $f^+(x)$ and the negative variation $f^-(x)$ of $f(x)$, they satisfy the conditions of Theorem 1.5. Therefore we define the positive LS-measures μ^+ and μ^- corresponding to $f^+(x)$ and $f^-(x)$ respectively. Then, if we put

$$\mu = \mu^+ - \mu^-,$$

we can define the LS-measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ and it is evident that this satisfies the condition in Theorem 1.8.

The inverse can be proved similarly.

In the similar way as in Theorem 1.8, we can characterize the LS-measure space by using a variable-wise right continuous function of locally bounded variation.

Theorem 1.9 (Existence theorem of a LS-measure) *Assume that a real-valued function $f(x)$ defined on $\boldsymbol{R}^d$ satisfies the following conditions* (i) *and* (ii):

(i) *$f(x)$ is a variable-wise right continuous function of locally bounded variation.*

(ii) *The two functions $f^+(x)$ and $f^-(x)$ are the positive variation and the negative variation of $f(x)$ respectively. We have the equality*

$$f(x) = f^+(x) - f^-(x).$$

Then there exists one and only one LS-*measure space* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ *such that we have the condition* (1) *in the following*:

(1) *For an interval E in the formula* (1.5), *we have the equality*

$$\mu(E) = \Delta_E f(x).$$

Further the inverse is also true.

Remark 1.1 We see that the positive LS-measure on $\boldsymbol{R}^d$ corresponding to the function $f(x) = x_1 x_2 \cdots x_d$ is the d-dimensional Lebesgue measure. Further the positive LS-measure on $\boldsymbol{R}^d$ corresponding to the function which is equal to $f(x) = x_1 x_2 \cdots x_d$ except in a multiplier by an arbitrary positive constant is the d-dimensional Lebesgue-Haar measure.

Theorem 1.10 *Assume that μ is a* LS-*measure on $\boldsymbol{R}^d$. Then there exist two positive* LS-*measures μ^+ and μ^- on $\boldsymbol{R}^d$ such that we have the unique expression*

$$\mu = \mu^+ - \mu^-.$$

We say that the result of Theorem 1.10 is the **Jordan decomposition** of μ.

Theorem 1.11 *Assume that μ is a* LS-*measure on $\boldsymbol{R}^d$ and λ is the* Lebesgue *measure on $\boldsymbol{R}^d$. Further assume that $f(x)$ is a continuous function of locally bounded variation which defines the* LS-*measure μ. Then $f(x)$ is absolutely continuous if and only if μ is absolutely continuous with respect to λ*

If $f(x)$ is absolutely continuous, $f(x)$ is continuous. Thus we assume that $f(x)$ is a continuous function of locally bounded variation as the precondition in Theorem 1.11.

2 Fundamental properties of the d-dimensional LS-measurable sets

In this section, we study the fundamental properties of the d-dimensional Lebesgue-Stieltjes measurable sets in the Lebesgue-Stieltjes measure space defined on the d-dimensional Euclidean space.

As the abbreviation of the expression, we say that the d-dimensional Lebesgue-Stieltjes measure defined on the d-dimensional Euclidean space is the d-dimensional LS-measure.

Further, as the abbreviation of the expression, we say that a d -dimensional Lebesgue-Stierltjes measurable set is a d-dimensional LS-measurable set.

In the sequel in this section, we have often to omit the adjective "d-dimensional".

Here, by characterizing the d-dimensional LS-measure defined in Definition 1.2, we prove that the family $\mathcal{M}$ of all LS-measurable sets is a σ-ring of sets and we study its relation to the LS-measure μ.

By restricting the LS-measure μ in Definition 1.2 to the family $\mathcal{R}$ of all blocks of intervals in $\boldsymbol{R}^d$, we have the concept of the LS-measure of the blocks of intervals in the following.

As the result, we see that the LS-measure of a block of intervals coincides with the RS-measure of the block of intervals.

Definition 2.1 Assume that $\mathcal{R}$ is the ring of all blocks of intervals in $\boldsymbol{R}^d$. Then we define that a set function μ on $\mathcal{R}$ is a **LS-measure** of blocks of intervals in $\boldsymbol{R}^d$ if we have the conditions (i) $\sim$ (iii) in the following:

(i) We have either one of the conditions (a) or (b) in the following:

(a) For an arbitrary set $A \in \mathcal{R}$, we have $-\infty < \mu(A) \leq \infty$.

(b) For an arbitrary set $A \in \mathcal{R}$, we have $-\infty \leq \mu(A) < \infty$.

(ii) If at most countable number of elements $A_1,\ A_2,\ \cdots,\ A_p,\ \cdots$ in $\mathcal{R}$ are mutually disjoint and their direct sum

$$A = \bigcup_{p=1}^{(\infty)} A_p = \sum_{p=1}^{(\infty)} A_p$$

is an element of $\mathcal{R}$, we have the equality

$$\mu(A) = \sum_{p=1}^{(\infty)} \mu(A_p).$$

(iii) For $A \in \mathcal{R}$, we have the equalities in the following:

$$\mu(A) = \mu^+(A) - \mu^-(A),$$

$$\nu(A) = \mu^+(A) + \mu^-(A).$$

Here we assume that ν is the total variation of μ, and μ^+ and μ^- are the positive variation and the negative variation of μ respectively.

Then we say that the value $\mu(E)$ of $E \in \mathcal{R}$ is the **LS-measure** of the block of intervals E.

Here we can see that this $\mu(E)$ coincides with the RS-measure of E

Corollary 2.1 *For a* LS-*measure* μ *of the blocks of intervals in* $\boldsymbol{R}^d$, *we have the statements* (1) $\sim$ (4) *in the following*:

(1) *If the elements* $A_1,\ A_2,\ \cdots,\ A_n$ *of* $\mathcal{R}$ *are mutually disjoint, we have the condition*

$$A = \bigcup_{p=1}^{n} A_p = \sum_{p=1}^{n} A_p \in \mathcal{R}$$

and we have the equality

$$\mu(A) = \sum_{p=1}^{n} \mu(A_p).$$

(2) *If we have* $A,\ B \in \mathcal{R}$ *and* $A \supset B$, *we have* $\mu^\pm(A) \geq \mu^\pm(B)$ *and* $\nu(A) \geq \nu(B)$. *Especially, if we have* $\nu(B) < \infty$, *we have* $\mu(A \backslash B) = \mu(A) - \mu(B)$. *Especially we have* $\mu(\emptyset) = 0$.

(3) *If, for at most countable number of elements* $A_1,\ A_2,\ \cdots,\ A_p,\ \cdots$ *of* $\mathcal{R}$, *we have the condition*

$$A = \bigcup_{p=1}^{(\infty)} A_p \in \mathcal{R},$$

we have the inequality

$$\lambda(A) \leq \sum_{p=1}^{(\infty)} \lambda(A_p).$$

Here λ *denotes one of the three measures* $\nu,\ \mu^+$ *and* μ^-.

(4) *If at most countable number of intervals* $I_1,\ I_2,\ \cdots,\ I_p,\ \cdots$ *are mutually disjoint and their direct sum*

$$I = \bigcup_{p=1}^{(\infty)} I_p = \sum_{p=1}^{(\infty)} I_p$$

is also an interval, we have the equality

$$\mu(I) = \sum_{p=1}^{(\infty)} \mu(I_p).$$

Proposition 2.1 *Assume that* μ *is a* LS-*measure of a block of intervals* E *in* $\mathcal{R}$. *Then, for a division of* E

$$E = I_1 + I_2 + \cdots + I_n \tag{2.4}$$

by virtue of the mutually disjoint intervals $I_1,\ I_2,\ \cdots,\ I_n$, *we have the equality*

$$\mu(E) = \mu(I_1) + \mu(I_2) + \cdots + \mu(I_n). \tag{2.5}$$

Here the value of the formula (2.5) *does not depend on the choice of a division by virtue of intervals in* E.

Inversely, we have the following theorem concerning the existence of a LS-measure of the blocks of intervals.

We have only to prove the existence of μ defined in Definition 2.1.

Theorem 2.1 *Assume that a real-valued function* $f(x)$ *is defined on* $\boldsymbol{R}^d$ *and it satisfies the conditions* (i) *and* (ii) *in the following*:

(i) $f(x)$ *is a variable-wise left continuous function.*

(ii) *For an arbitrary interval*

$$E = \prod_{p=1}^{d} I_p, \tag{2.6}$$

the condition

$$\Delta_E f(x) \geq 0$$

is satisfied. Here we denote

$$I_p = [x_p,\ y_p) \text{ or } I_p = \{x_p\} = [x_p,\ x_p + 0),$$

$$(x_p,\ y_p \in \boldsymbol{R},\ x_p < y_p,\ (1 \leq p \leq d)).$$

Then there exists one and only one positive LS-*measure* μ *on* $\mathcal{R}$ *such that it satisfies the conditions* (1) $\sim$ (3) *in the following*:

(1) *For an interval* E *of the formula* (2.6), *we have the equality*

$$\mu(E) = \Delta_E f(x).$$

Here we use the same notation as in Lemma 1.2.

(2) *For a division of a block of intervals* E

$$E = E_1 + E_2 + \cdots + E_n$$

by using the intervals $E_1,\ E_2,\ \cdots,\ E_n$ *in* E, *we have the equality*

$$\mu(E) = \mu(E_1) + \mu(E_2) + \cdots + \mu(E_n).$$

Further, the value of $\mu(E)$ *is determined uniquely and independent to the choice of a division of* E *by virtue of intervals.*

(3) *If at most countable number of intervals* $E_1,\ E_2,\ \cdots,\ E_n,\ \cdots$ *are mutually disjoint and its direct sum*

$$E = \bigcup_{p=1}^{(\infty)} E_p = \sum_{p=1}^{(\infty)} E_p$$

is also an interval, we have the equality

$$\mu(E) = \sum_{p=1}^{(\infty)} \mu(E_p).$$

Then $\mu(A)$ *is a positive* LS-*measure of a block of intervals* A.

Next, as for the definition and the existence theorem of a d-dimensional positive BS-measure, we refer to Definition 1.3 and Lemma 1.3.

At last, we determine the d-dimensional positive LS-measure μ concretely and we prove the existence theorem of the d-dimensional positive LS-measure.

Here we prepare the necessary facts for that purpose.

By virtue of Definition 1.2, we have only to prove the existence of a d-dimensional positive LS-measure. Therefore we assume that a BS-measure μ on $\boldsymbol{R}^d$ is positive.

Definition 2.2 For an arbitrary subset A of $\boldsymbol{R}^d$, we define that

$$\mu^*(A) = \inf\{\mu(B);\ B \supset A,\ B \in \boldsymbol{B}\},$$

$$\mu_*(A) = \sup\{\mu(B);\ A \supset B,\ B \in \boldsymbol{B}\}$$

are the **outer measure** and the **inner measure** of A respectively.

Corollary 2.2 *For* $A \in \boldsymbol{B}$, *we have the equality*

$$\mu^*(A) = \mu_*(A) = \mu(A).$$

Here the third side of this equality denotes the positive BS-*measure.*

By virtue of the definition of the outer measure and the inner measure, we have the three propositions in the following.

In the sequel, we assume that $A,\ A_1$ and A_2 are three subsets of $\boldsymbol{R}^d$.

Proposition 2.2 *We have* $0 \leq \mu_*(A) \leq \mu^*(A) \leq +\infty$. *Especially we have* $\mu^*(\phi) = \mu_*(\phi) = 0$.

Proposition 2.3 *If we have* $A_1 \subset A_2$, *we have the results* (1) *and* (2) *in the following*:

(1) $\mu^*(A_1) \leq \mu^*(A_2)$. (2) $\mu_*(A_1) \leq \mu_*(A_2)$.

Proposition 2.4 *We have the inequality in the following*:

$$\mu^*(A_1 \cup A_2) \leq \mu^*(A_1) + \mu^*(A_2).$$

Proposition 2.5 *If, for a countable number of subsets* $A_1,\ A_2,\ \cdots,\ A_p,\ \cdots$ *of* $\boldsymbol{R}^d$, *we put*

$$A = \bigcup_{p=1}^{\infty} A_p,$$

we have the inequality

$$\mu^*(A) \leq \sum_{p=1}^{\infty} \mu^*(A_p).$$

Proposition 2.6 *If a countable number of subsets* $A_1,\ A_2,\ \cdots,\ A_p,\ \cdots$ *of* $\boldsymbol{R}^d$ *are mutually disjoint and we put*

$$A = \sum_{p=1}^{\infty} A_p,$$

we have the inequality

$$\mu_*(A) \geq \sum_{p=1}^{\infty} \mu_*(A_p).$$

Proposition 2.7 *We assume that* A *is an arbitrary bounded set* $E \in \boldsymbol{B}$, *we have the equality*

$$\mu_*(A \cap E) = \mu(E) - \mu^*(A^c \cap E).$$

Here μ *is a positive* BS-*measure.*

Proposition 2.8 *Assume that* A *is an arbitrary subset in* $\boldsymbol{R}^d$. *Assume that* $E_1,\ E_2,\ \cdots$ *is a sequence of bounded* Borel *sets in* $\boldsymbol{R}^d$ *and they satisfy the conditions*

$$E_1 \subset E_2 \subset \cdots,\ \bigcup_{n=1}^{\infty} E_n = \boldsymbol{R}^d.$$

Then we have the equalities

$$\mu^*(A) = \lim_{n\to\infty} \mu^*(A \cap E_n), \tag{2.13}$$

$$\mu_*(A) = \lim_{n\to\infty} \mu_*(A \cap E_n). \tag{2.14}$$

Definition 2.3 We use the notation in Definition 2.2. We define that an arbitrary set A in $\boldsymbol{R}^d$ is **LS-measurable** if, for an arbitrary bounded set $E \in \boldsymbol{B}$, we have the equality

$$\mu^*(A \cap E) = \mu_*(A \cap E).$$

Then we say that

$$\mu(A) = \sup\{\mu^*(A \cap E);\ E \in \boldsymbol{B} \text{ is bounded}\}$$

is the **positive LS-measure** of A.

We denote the family of all LS-measurable sets in $\boldsymbol{R}^d$ as $\mathcal{M}$.

Remark 2.1 In Definition 2.3, a subset A in $\boldsymbol{R}^d$ is LS-measurable if and only if the outer measure $\mu^*(A \cap E)$ and the inner measure $\mu_*(A \cap E)$ coincide for any bounded set $E \in \boldsymbol{B}$. Here, $\mu^*(A \cap E)$ is the approximation of a bounded part $A \cap E$ of A by using the measures of bounded Borel sets from the outer side and $\mu_*(A \cap E)$ is the approximation of a bounded part $A \cap E$ of A by using the measures of bounded Borel sets from the inner side.

Corollary 2.3 *We use the notation in* Definition 2.3. *Then, if A is an arbitrary* LS-*measurable set in $\boldsymbol{R}^d$, we have the equality*

$$\mu^*(A) = \mu_*(A) = \mu(A).$$

In the sequel, we prove that the set function μ defined in Definition 2.3 satisfies the conditions of the positive LS-measure in Definition 1.2. Namely we prove that $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ is a positive LS-measure space.

In Corollary 2.2, we can see that the LS-measure of a Borel set coincides with the BS-measure of the Borel set.

Theorem 2.2 *Assume that A is an arbitrary set in $\boldsymbol{R}^d$. Then A is* LS-*measurable if and only if, for an arbitrary set $E \in \boldsymbol{B}$, we have the equality*

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu(E).$$

Theorem 2.3 *Assume that A is an arbitrary set in $\boldsymbol{R}^d$. Then A is* LS-*measurable if and only if, for an arbitrary set B in $\boldsymbol{R}^d$, we have the equality*

$$\mu^*(A \cap B) + \mu^*(A^c \cap B) = \mu^*(B).$$

Theorem 2.4 *Assume that A is an arbitrary set in $\boldsymbol{R}^d$. Then A is* LS-*measurable if and only if, for two arbitrary sets A_1 and A_2 such that we have the conditions $A_1 \subset A$ and $A_2 \subset A^c$, we have the equality*

$$\mu^*(A_1 + A_2) = \mu^*(A_1) + \mu^*(A_2).$$

Theorem 2.5 *Assume that A is an arbitrary bounded set in $\boldsymbol{R}^d$. Then A is* LS-*measurable if and only if, for an arbitrary $\varepsilon > 0$, there exist two sets $A_1,\ A_2 \in \boldsymbol{B}$ such that we have the conditions* (1) *and* (2) *in the following*:

(1) *We have $A_1 \subset A \subset A_2$.*

(2) *We have* $\mu(A_2 \backslash A_1) < \varepsilon$.

Here μ *is assumed to be the positive* BS-*measure on* $\boldsymbol{B}$.

Theorem 2.6 *We assume that* $\mathcal{O}$ *is the family of all open sets in* $\boldsymbol{R}^d$, $\mathcal{C}$ *is the family of all closed sets in* $\boldsymbol{R}^d$ *and* $\mathcal{M}$ *is the family of all* LS-*measurable sets in* $\boldsymbol{R}^d$. *Then we have the statements* $(1) \sim (3)$ *in the following*:

(1) *We have* $\mathcal{O} \cup \mathcal{C} \subset \boldsymbol{B} \subset \mathcal{M}$. *Especially we have* $\phi \in \mathcal{M}$.

(2) *If we have* $A \in \mathcal{M}$, *we have* $A^c \in \mathcal{M}$.

(3) *If we have* $A,\ B \in \mathcal{M}$, *we have* $A \cup B \in \mathcal{M}$.

Corollary 2.4 *Assume that* $\mathcal{M}$ *is the same as in* Theorem 2.6. *Then we have the following*:

(1) *We have* $\boldsymbol{R}^d \in \mathcal{M}$.

(2) *If we have* $A,\ B \in \mathcal{M}$, *we have* $A - B \in \mathcal{M}$.

(3) *For* $A_p \in \mathcal{M},\ (p = 1,\ 2,\ \cdots,\ n)$, *we have the following*

$$\bigcup_{p=1}^{n} A_p \in \mathcal{M},\ \bigcap_{p=1}^{n} A_p \in \mathcal{M}.$$

Therefore $\mathcal{M}$ *is an algebra of sets in* $\boldsymbol{R}^d$.

Theorem 2.7 *If we have* $A,\ B \in \mathcal{M}$ *and* $A \cap B = \phi$, *we have the equality*

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Theorem 2.8 *If a countable number of sets* $A_1,\ A_2, \cdots,\ A_n,\ \cdots$ *in* $\mathcal{M}$ *are mutually disjoint, we have the condition*

$$A = \bigcup_{p=1}^{\infty} A_p = \sum_{p=1}^{\infty} A_p \in \mathcal{M}$$

and we have the equality

$$\mu(A) = \sum_{p=1}^{\infty} \mu(A_p).$$

Corollary 2.5 *For* $A_p \in \mathcal{M},\ (p \geq 1)$, *we have the condition in the following*:

$$\bigcap_{p=1}^{\infty} A_p \in \mathcal{M}.$$

Therefore we have the theorem in the following.

Theorem 2.9 *The measure space* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ *is the d-dimensional positive* LS-*measure space where* $\mathcal{M}$ *is the* σ-*algebra of all* LS-*measurable sets defined in* Definition 2.3 *and the set function* μ *is the d-dimensional positive* LS-*measure.*

Since we prove that the measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ in Theorem 2.9 satisfies the system of axioms of the d-dimensional positive LS-measure space in Definition 1.2, we prove the existence theorem of the d-dimensional positive LS-measure space.

Simultaneously we prove the existence theorem of a general d-dimensional LS-measure space. Namely we have the theorem in the following.

Theorem 2.10 *There exists a d-dimensional* LS-*measure space* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$.

By virtue of the definition of the LS-measure, we see that a RS-measurable set is a LS-measurable set and the value of the RS-measure and the value of the LS-measure for a RS-measurable set are identical.

We assume that $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ is a general LS-measure space and ν is the total variation of μ.

Then we define that a subset e of $\boldsymbol{R}^d$ is a **null set** if we have the outer measure $\nu^*(e) = 0$. The empty set ϕ is a null set. Inversely an arbitrary null set need not be the empty set.

Proposition 2.9 *A null set e is* LS-*measurable and we have* $\mu(e) = 0$.

The null sets have the following properties.

Proposition 2.10 *We have the statements* (1) *and* (2) *in the following*:

(1) *A subset of a null set is a null set.*

(2) *The union of at most countable number of null sets* $e_1,\ e_2, \cdots$

$$e = \bigcup_{p=1}^{(\infty)} e_p$$

is also a null set.

Next, when $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ is a general LS-measure space, we study the fundamental properties of the LS-measurable sets and the LS-measure. Especially the relation of a limit set and its measure is important. Since the LS-measure is completely additive, it is characteristic that we can very well calculate the measure of a limit set.

Then the study of the inequality of the measures is fundamental. Therefore, in the sequel in this chapter, we study the case of a positive LS-measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$.

In the case of a general LS-measure space $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$, the results of those studies on the total variation ν, the positive variation μ^+ and the negative variation μ^- of μ are meaningful.

Then, because the family of all LS-measurable sets $\mathcal{M}$ is a σ-algebra, we have the proposition in the following.

Proposition 2.11 *We have the statements* (1) *and* (2) *in the following*:

(1) *If we have* $A_1,\ A_2,\ \cdots \in \mathcal{M}$, *we have* $\overline{\lim_{p\to\infty}}\ A_p,\ \underline{\lim_{p\to\infty}}\ A_p \in \mathcal{M}$.

(2) *If we have* $\lim_{p\to\infty} A_p$, *we have* $\lim_{p\to\infty} A_p \in \mathcal{M}$.

Theorem 2.11 *Assume that* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ *is a positive* LS-*measure space. Then, if we have the condition* $A_1,\ A_2,\ \cdots \in \mathcal{M}$, *we have the statements* (1) $\sim$ (4) *in the following*:

(1) *If we have either one of the conditions* (i) *and* (ii) *in the following*:

  (i) *we have* $A_1 \subset A_2 \subset \cdots$,

  (ii) *We have* $A_1 \supset A_2 \supset \cdots$ *and* $\mu(A_1) < \infty$,

we have the equality

$$\mu(\lim_{p\to\infty} A_p) = \lim_{p\to\infty} \mu(A_p).$$

(2) *We have the inequality*

$$\mu(\underline{\lim_{p\to\infty}}\ A_p) \leq \underline{\lim_{p\to\infty}}\ \mu(A_p).$$

(3) *If we have the condition*

$$\mu(\bigcup_{p=1}^{\infty} A_p) < \infty,$$

we have the inequality

$$\mu(\overline{\lim_{p\to\infty}}\ A_p) \geq \overline{\lim_{p\to\infty}}\ \mu(A_p).$$

(4) *If we have the condition*

$$\mu(\bigcup_{p=1}^{\infty} A_p) < \infty$$

and we have

$$\lim_{p\to\infty} A_p,$$

we have the equality

$$\mu(\lim_{p\to\infty} A_p) = \lim_{p\to\infty} \mu(A_p).$$

Theorem 2.12 *Assume that* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ *is a positive* LS-*measure space. Assume that* A *is an arbitrary bounded set in* $\boldsymbol{R}^d$ *and it is not necessarily measurable. Then, for an arbitrary* $\varepsilon > 0$, *there exist an open set* G *and a closed set* F *such that we have the following*:

$$A \subset G \text{ and } \mu(G) < \mu^*(A) + \varepsilon,$$

$$F \subset A \text{ and } \mu(F) > \mu_*(A) - \varepsilon.$$

Theorem 2.13 *Assume that* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ *is a positive* LS-*measure space. If* A *is a* LS-*measurable set in* $\boldsymbol{R}^d$, *then, for an arbitrary* $\varepsilon > 0$, *there exist an open set* G *and a closed set* F *such that we have the conditions in the following*:

$$F \subset A \subset G,\ \mu(G\backslash A) < \varepsilon,\ \mu(A\backslash F) < \varepsilon.$$

Especially, if we have $\mu(A) < \infty$, *we can obtain* F *as a bounded closed set.*

Corollary 2.6 *We use the notation in* Theorem 2.13. *If* A *is a* LS-*measurable set, then there exists a* Borel *set* B *such that we have the conditions in the following*:

$$A \subset B,\ \mu(B\backslash A) = 0.$$

Corollary 2.7 *We use the notation in* Theorem 2.13. *If* A *is a* LS-*measurable set, then there exists a* Borel *set* B *such that we have the conditions in the following*:

$$B \subset A,\ \mu(A\backslash B) = 0.$$

By virtue of these Corollaries, a LS-measurable set is expressed as a difference set or a union of a Borel set and a null set.

We assume that $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ is a positive LS-measure space.
Now, for two sets A and B, we put

$$A\Delta B = (A\backslash B) + (B\backslash A).$$

Then, we say that a sequence of sets $\{A_n\}$ **converges** to A **in measure** if we have the condition

$$\mu^*(A_n \Delta A) \to 0,\ (n \to \infty).$$

Theorem 2.14 *Assume that* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ *is a positive* LS-*measure space. Then a bounded set* A *is* LS-*measurable if and only if there exists a sequence of bounded* Borel *sets* $\{A_n\}$ *such that we have the condition*

$$\mu^*(A_n \Delta A) \to,\ (n \to \infty).$$

Then we have the equality

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

Theorem 2.15 *Assume that* $(\boldsymbol{R}^d,\ \mathcal{M},\ \mu)$ *is a positive* LS-*measure space. Assume that* A *is a bounded set in* $\boldsymbol{R}^d$ *and* $\{A_n\}$ *is a sequence of bounded* LS-*measurable sets. Then, if we have the condition*

$$\mu^*(A_n \Delta A) \to 0,\ (n \to \infty),$$

A *is* LS-*measurable.*

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