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Decay Properties for Mildly Degenerate Kirchhoff Type Dissipative Wave Equations in Bounded Domains

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Abstract

Under the assumption that the initial data belong to suitable Sobolev spaces, we derive the better decay estimate of the second order derivatives for the initial boundary value problem for degenerate dissipative wave equations of Kirchhoff type.

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1 Introduction

In this paper, we study on the decay rates of solutions to the initial boundary value problem for the following degenerate dissipative wave equations of Kirchhoff type :

$$\begin{cases} \rho u'' + \|A^{1/2}u(t)\|^{2\gamma} Au + u' = 0 \quad \text{in} \quad \Omega \times [0,\infty)\,, \\ u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x) \quad \text{in} \quad \Omega\,, \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times [0,\infty)\,, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is an unknown real value function, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $' = \partial/\partial t$, $A = -\Delta = -\sum_{j=1}^{N} \partial^2/\partial x_j^2$ is the Laplace operator with the domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $\|\cdot\|$ is the norm of $L^2(\Omega)$, and $\rho > 0$ and $\gamma > 0$ are positive constants.

It is well known that Equation (1.1) describes the damped small amplitude vibrations of an elastic, stretched string when the dimension N is one or membrane when the dimension N is two (see Kirchhoff [7] and Carrier [3]).

The unique global solvability has been considered for the initial data $[u_0, u_1]$ belonging to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ and $\|A^{1/2}u_0\| \neq 0$ (cf. [1], [2], [13] for local solvability). When $\gamma \geq 1$, under the assumption that the initial data $[u_0, u_1]$ are small Nishihara and Yamada [12] have shown global existence theorems and they derived some decay estimates such that

$$\|A^{1/2}u(t)\|^2 \leq C(1+t)^{-\frac{1}{\gamma}}, \quad \|A^{1/2}u'(t)\|^2 \leq C(1+t)^{-1}, \tag{1.2}$$

$$\|u'(t)\|^2 + \|u''(t)\|^2 \leq C(1+t)^{-1-\frac{1}{\gamma}} \quad \text{for} \quad t \geq 0. \tag{1.3}$$

When $\gamma = 1$, in the previous paper [13], we improved the decay rates (1.2)–(1.3) as in the upper estimates (1.4)–(1.6) for $\gamma = 1$. (see Nishihara [11], Mizumachi [8], Ono [14] for lower decay estimates). When $\gamma > 0$, under the assumption that the coefficient $\rho > 0$ is small, Ghisi and Gobbino [5] have derived some decay estimates such that

$$C'(1+t)^{-\frac{1}{\gamma}} \leq \|A^{m/2}u(t)\|^2 \leq C(1+t)^{-\frac{1}{\gamma}} \qquad \text{for} \quad m = 1, 2.$$

Finally, when $\gamma > 0$, in previous paper [15], under the assumption that the coefficient $\rho > 0$ or the initial data $[u_0, u_1] \in \mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ are small, we have derived the decay estimates such that

$$C'(1+t)^{-\frac{1}{\gamma}} \leq \|A^{k/2}u(t)\|^2 \leq C(1+t)^{-\frac{1}{\gamma}} \qquad \text{for} \quad k = 0, 1, 2, \tag{1.4}$$

$$\|A^{j/2}u'(t)\|^2 \leq C(1+t)^{-2-\frac{1}{\gamma}} \qquad \text{for} \quad j = 0, 1, \tag{1.5}$$

$$\|u''(t)\|^2 \leq C(1+t)^{-3-\frac{1}{\gamma}} \qquad \text{for} \quad t \geq 0 \tag{1.6}$$

(see [14] for $\gamma = 1$). However the decay rate of the estimate (1.6) is not optimal.

In this paper, we discuss to derive the better decay rate of the norm $\|u''(t)\|^2$ under an additional assumption on the initial data $[u_0, u_1]$ (see Ghisi [4] for the similar decay rate together with a different analisys).

Our main result is as follows.

Theorem 1.1 *Let the initial data* $[u_0, u_1]$ *belong to* $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ *and* $\|A^{1/2}u_0\| \neq 0$. *Suppose that the coefficient* $\rho > 0$ *or the initial energy* $E(0)$ *is small in the sense of* (2.5). *Then, the problem* (1.1) *admits a unique global solution* $u(t)$ *in the class* $C^0([0,\infty); \mathcal{D}(A)) \cap C^1([0,\infty); \mathcal{D}(A^{1/2})) \cap C^2([0,\infty); L^2(\Omega))$ *and this solution* $u(t)$ *has the decay properties* (1.4)–(1.6).

Moreover, if the initial data $[u_0, u_1] \in \mathcal{D}(A^{3/2}) \times \mathcal{D}(A)$, *then it holds that*

$$C'(1+t)^{-\frac{1}{\gamma}} \leq \|A^{k/2}u(t)\|^2 \leq C(1+t)^{-\frac{1}{\gamma}} \quad \textit{for} \quad k = 0, 1, 2, 3, \tag{1.7}$$

$$\|A^{j/2}u'(t)\|^2 \leq C(1+t)^{-2-\frac{1}{\gamma}} \quad \textit{for} \quad j = 0, 1, 2, \tag{1.8}$$

$$\|u''(t)\|^2 \leq C(1+t)^{-4-\frac{1}{\gamma}} \quad \textit{for} \quad t \geq 0, \tag{1.9}$$

where C *and* C' *are some positive constants.*

Theorem 1.1 follows from Theorem 2.1 and Propositions 3.1 – 3.3 and Theorem 3.4 in the continuing sections. The notations we use in this paper are standard. The symbol $(\cdot,\cdot)$ means the inner product in $L^2(\Omega)$ or sometimes duality between the space X and its dual X'. Positive constants will be denoted by C and will change from line to line.

2 Preliminaries

We introduce an energy $E(t)$ as

$$E(t) \equiv \rho\|u'(t)\|^2 + \frac{1}{\gamma+1}M(t)^{\gamma+1} \qquad \text{with} \quad M(t) \equiv \|A^{1/2}u(t)\|^2 . \tag{2.1}$$

By simple calculation, we see that the energy $E(t)$ has the so-called energy identity such that

$$\frac{d}{dt}E(t) + 2\|u'(t)\|^2 = 0 \tag{2.2}$$

or

$$E(t) + 2\int_0^t \|u'(s)\|^2\,ds = E(0)\,. \tag{2.3}$$

Moreover, we will use the function $H(t)$ (a second order energy) as

$$H(t) \equiv \rho\frac{\|A^{1/2}u'(t)\|^2}{M(t)^\gamma} + \|Au(t)\|^2\,. \tag{2.4}$$

In previous paper [16], we have proved the following global existence theorem and obtained some decay properties.

Theorem 2.1 *Let the initial data* $[u_0, u_1]$ *belong to* $\mathcal{D}(A)\times\mathcal{D}(A^{1/2})$ *and* $M(0) > 0$. *Suppose that*

$$2(\gamma+1)^{\frac{2\gamma+1}{\gamma+1}}G(0)^{\frac12}B(0)^{\frac12}\rho E(0)^{\frac{\gamma}{\gamma+1}} < 1 \tag{2.5}$$

Then, the problem (1.1) *admits a unique global solution* $u(t)$ *in the class*

$$C^0([0,\infty);\mathcal{D}(A)) \cap C^1([0,\infty);\mathcal{D}(A^{1/2})) \cap C^2([0,\infty);L^2(\Omega))\,,$$

and this solution $u(t)$ *satisfies*

$$\rho\frac{|M'(t)|}{M(t)} < \frac{1}{\gamma+1} \quad \textit{and} \quad H(t) \le H(0)\,, \tag{2.6}$$

$$\frac{\|Au(t)\|^2}{M(t)} \le G(0) \quad \textit{and} \quad \frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} \le B(0)\,, \tag{2.7}$$

$$C'(1+t)^{-\frac{1}{\gamma}} \le M(t) \le ((\gamma+1)E(0))^{\frac{1}{\gamma+1}} \quad \textit{for} \quad t \ge 0\,, \tag{2.8}$$

where

$$E(0) \equiv \rho\|u_1\|^2 + \frac{1}{\gamma+1}\|A^{1/2}u_0\|^2\,, \tag{2.9}$$

$$H(0) \equiv \rho\frac{\|A^{1/2}u_1\|^2}{\|A^{1/2}u_0\|^{2\gamma}} + \|Au_0\|^2\,, \tag{2.10}$$

$$G(0) \equiv \frac{\|Au_0\|^2}{\|A^{1/2}u_0\|^2} + \rho\left(\frac{\|A^{1/2}u_1\|^2}{\|Au_0\|^{2\gamma}} - \frac{(A^{1/2}u_1, A^{1/2}u_0)}{2\|A^{1/2}u_0|^{2\gamma+2}}\right)\,, \tag{2.11}$$

$$B(0) \equiv \max\left\{\frac{\|u_1\|^2}{\|A^{1/2}u_0\|^{4\gamma+2}}\,,\,(2(\gamma+1))^2G(0)\right\}\,, \tag{2.12}$$

and C' is some positive constant.

In order to derive decay estimates of the solution $u(t)$ of (1.1), the following generalized Nakao type inequality is useful (see [6] and [15] for the proof and also see [9], [10], [17]).

Lemma 2.2 *Let $\phi(t)$ be a non-negative function and satisfy*

$$\sup_{t\le s\le t+1}\phi(s)^{1+\alpha} \le (k_0\phi(t)^\alpha + k_1(1+t)^{-\beta})(\phi(t)-\phi(t+1)) + k_2(1+t)^{-\gamma}$$

with certain constants k_0, k_1, $k_2 \ge 0$, $\alpha > 0$, $\beta \ge 0$, and $\gamma > 0$. Then, the function $\phi(t)$ satisfies

$$\phi(t) \le C_0(1+t)^{-\theta}\,, \quad \theta = \min\left\{\frac{1+\beta}{\alpha}, \frac{\gamma}{1+\alpha}\right\}$$

for $t \ge 0$ with some constant C_0 depending on $\phi(0)$.

3 Decay Estimetes

By the same analysis as in previous paper [15], using the estimates (2.6)–(2.8), we can obtain the following decay estimates (or (1.4) and (1.5)). We omit the proof here (see [15]).

Proposition 3.1 *Under the assumption of Theorem 2.1, it holds that*

$$C'(1+t)^{-\frac{1}{\gamma}} \le \|A^{k/2}u(t)\|^2 \le C(1+t)^{-\frac{1}{\gamma}} \quad \textit{for} \quad k=0,1,2\,, \tag{3.1}$$

$$\|u'(t)\|^2 \le C(1+t)^{-2-\frac{1}{\gamma}} \quad \textit{for} \quad t \ge 0\,, \tag{3.2}$$

where C and C' are some positive constants.

Proposition 3.2 *Under the assumption of Theorem 2.1, if the initial data* $[u_0, u_1]$ *belong to* $\mathcal{D}(A^{3/2}) \times \mathcal{D}(A)$*, then it holds that*

$$F(t) \equiv \rho \frac{\|A^{1/2}u''(t)\|^2}{M(t)^\gamma} + \|Au'(t)\|^2 \le C(1+t)^{-2-\frac{1}{\gamma}} \tag{3.3}$$

and

$$\|A^{1/2}u''(t)\|^2 \le C(1+t)^{-3-\frac{1}{\gamma}} \quad \text{for} \quad t \ge 0\,. \tag{3.4}$$

Proof. Differentiating Equation (1.1) once with respect to t, we have

$$\rho u''' + M(t)^\gamma Au' + \gamma \frac{M'(t)}{M(t)} M(t)^\gamma Au + u'' = 0\,. \tag{3.5}$$

Multiplying (3.5) by $2M(t)^{-\gamma}Au''$ over Ω and integrating it over Ω, we have from Equation (1.1) that

$$\begin{aligned} \frac{d}{dt}F(t) + 2\left(1 + \frac{\gamma}{2}\rho \frac{M'(t)}{M(t)}\right) \frac{\|A^{1/2}u''(t)\|^2}{M(t)^\gamma} &= 2\gamma \frac{M'(t)}{M(t)} \frac{(A^{1/2}u'(t), A^{1/2}u''(t))}{M(t)^\gamma} \\ &\le 4\gamma \frac{\|A^{1/2}u'(t)\|^2 \|A^{1/2}u''(t)\|}{M(t)^{\gamma+\frac{1}{2}}}\,. \end{aligned} \tag{3.6}$$

Since it follows from (2.6) that

$$1 + \frac{\gamma}{2}\rho \frac{M'(t)}{M(t)} \ge \frac{\gamma+2}{2(\gamma+1)} > \frac{1}{2}\,,$$

the Young inequality yields

$$\frac{d}{dt}F(t) + \frac{\|A^{1/2}u''(t)\|^2}{M(t)^\gamma} \le Cf(t)^2 \qquad \text{with} \quad f(t)^2 \equiv \frac{\|A^{1/2}u'(t)\|^4}{M(t)^{\gamma+1}}\,. \tag{3.7}$$

Integrating (3.7) over $[t, t+1]$, we have

$$\int_t^{t+1} \frac{\|A^{1/2}u''(s)\|^2}{M(s)^\gamma}\,ds \le F(t) - F(t+1) + C \sup_{t \le s \le t+1} f(s)^2 \qquad \left(\equiv D(t)^2\right)\,. \tag{3.8}$$

Then, there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\frac{\|A^{1/2}u''(t_j)\|^2}{M(t_j)^\gamma} \le 4D(t)^2 \qquad \text{for} \quad j = 1, 2\,. \tag{3.9}$$

Multiplying (3.5) by $M(t)^{-\gamma}Au'$ and integrating it over Ω, we have from Equation (1.1) that

$$\begin{aligned}\|Au'(t)\|^2 &= \rho\frac{\|A^{1/2}u''(t)\|^2}{M(t)^\gamma} - \rho\frac{d}{dt}\frac{(A^{1/2}u''(t), A^{1/2}u'(t))}{M(t)^\gamma} \\ &\quad - \frac{(A^{1/2}u''(t), A^{1/2}u'(t))}{M(t)^\gamma} + \gamma\frac{M'(t)}{M(t)}\frac{\|A^{1/2}u'(t)\|^2}{M(t)^\gamma}.\end{aligned} \tag{3.10}$$

Integrating (3.10) over $[t_1, t_2]$, we observe

$$\begin{aligned}\int_{t_1}^{t_2}\|Au'(s)\|^2\,ds &\le \rho\int_t^{t+1}\frac{\|A^{1/2}u''(s)\|^2}{M(s)^\gamma}\,ds + \rho\sum_{j=1}^{2}\frac{\|A^{1/2}u''(t_j)\|\|A^{1/2}u'(t_j)\|}{M(t_j)^\gamma} \\ &\quad + \int_t^{t+1}\frac{\|A^{1/2}u''(s)\|\|A^{1/2}u'(s)\|}{M(s)^\gamma}\,ds + 2\gamma\int_t^{t+1}\frac{\|A^{1/2}u'(s)\|^3}{M(s)^{\gamma+\frac{1}{2}}}\,ds \\ &\le D(t)^2 + CD(t)\sup_{t\le s\le t+1} g(s) + C\sup_{t\le s\le t+1} h(s)^2\end{aligned}$$

with

$$g(t)^2 \equiv \frac{\|A^{1/2}u'(t)\|^2}{M(t)^\gamma} \qquad \text{and} \qquad h(t)^2 \equiv \frac{\|A^{1/2}u'(t)\|^3}{M(t)^{\gamma+\frac{1}{2}}}, \tag{3.11}$$

and moreover,

$$\begin{aligned}\int_{t_1}^{t_2} F(s)\,ds &= \rho\int_{t_1}^{t_2}\frac{\|A^{1/2}u''(s)\|^2}{M(s)^\gamma}\,ds + \int_{t_1}^{t_2}\|Au'(s)\|^2\,ds \\ &\le CD(t)^2 + CD(t)\sup_{t\le s\le t+1} g(s) + C\sup_{t\le s\le t+1} h(s)^2.\end{aligned} \tag{3.12}$$

There exists $t_* \in [t_1, t_2]$ such that

$$F(t_*) \le 2\int_{t_1}^{t_2} F(s)\,ds. \tag{3.13}$$

For $\tau \in [t, t+1]$, integrating (3.6) over $[\tau, t_*]$ (or $[t_*, \tau]$), we have from the Young inequality that

$$\begin{aligned}F(\tau) &= F(t_*) - \int_{t_*}^{\tau}\left(2 - \gamma\rho\frac{M'(s)}{M(s)}\right)\frac{\|A^{1/2}u''(s)\|^2}{M(s)^\gamma}\,ds \\ &\quad + 2\gamma\int_{t_*}^{\tau}\frac{M'(s)}{M(s)}\frac{(A^{1/2}u'(s), A^{1/2}u''(s))}{M(s)^\gamma}\,ds \\ &\le F(t_*) + C\int_t^{t+1}\frac{\|A^{1/2}u''(s)\|^2}{M(s)^\gamma}\,ds + C\int_t^{t+1}\frac{\|A^{1/2}u'(s)\|^3}{M(s)^{\gamma+\frac{1}{2}}}\,ds,\end{aligned}$$

and from (3.8), (3.11), (3.12), and (3.13) that

$$\sup_{t\le s\le t+1} F(s) \le CD(t)^2 + CD(t) \sup_{t\le s\le t+1} g(s) + C \sup_{t\le s\le t+1} h(s)^2$$

or

$$\sup_{t\le s\le t+1} F(s)^2 \le C\left(D(t)^2 + \sup_{t\le s\le t+1} g(s)^2\right) D(t)^2 + C \sup_{t\le s\le t+1} h(s)^4 .$$

From (3.8) and the Young inequality, we observe

$$\sup_{t\le s\le t+1} F(s)^2 \le C\left(F(t) + \sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right)(F(t) - F(t+1))$$
$$+ C\left(\sup_{t\le s\le t+1} f(s)^2 + \sup_{t\le s\le t+1} g(s)^2\right) \sup_{t\le s\le t+1} f(s)^2 + C \sup_{t\le s\le t+1} h(s)^4 .$$

Since it follows from (2.8), (3.2), (3.7), and (3.11) that

$$f(t)^2 \equiv \frac{\|A^{1/2}u'(t)\|^4}{M(t)^{\gamma+1}} \le C(1+t)^{-3-\frac{1}{\gamma}} ,$$
$$g(t)^2 \equiv \frac{\|A^{1/2}u'(t)\|^2}{M(t)^{\gamma}} \le C(1+t)^{-1-\frac{1}{\gamma}} ,$$
$$h(t)^2 \equiv \frac{\|A^{1/2}u'(t)\|^3}{M(t)^{\gamma+\frac{1}{2}}} \le C(1+t)^{-2-\frac{1}{\gamma}} ,$$

we have

$$\sup_{t\le s\le t+1} F(s)^2 \le C\left(F(t) + (1+t)^{-1-\frac{1}{\gamma}}\right)(F(t) - F(t+1)) + C(1+t)^{-4-\frac{2}{\gamma}} , \tag{3.14}$$

and moreover, applying Lemma 2.2 to (3.14) we obtain the desired estimate (3.3). (3.4) follows from (3.3) and (3.1) with $k = 1$. □

Proposition 3.3 *Under the assumption of Proposition 3.2, it holds that*

$$\|u''(t)\|^2 \le C(1+t)^{-4-\frac{1}{\gamma}} , \tag{3.15}$$
$$\|Au(t)\|_{H^1}^2 \le C(1+t)^{-\frac{1}{\gamma}} \quad \textit{for} \quad t \ge 0 . \tag{3.16}$$

Proof. Multiplying (3.5) by $2u''$ and integrating it over Ω, we have

$$\rho\frac{d}{dt}\|u''(t)\|^2 + 2\|u''(t)\|^2$$
$$= -2M(t)^{\gamma}(Au'(t), u''(t)) - 2\gamma\frac{M'(t)}{M(t)}M(t)^{\gamma}(Au(t), u''(t))$$
$$\le CM(t)^{\gamma}\|Au'(t)\|\|u''(t)\| + C\|A^{1/2}u'(t)\|M(t)^{\gamma}\frac{\|Au(t)\|}{M(t)^{\frac{1}{2}}}\|u''(t)\| ,$$

and the Young inequality yields

$$\rho\frac{d}{dt}\|u''(t)\|^2+2\|u''(t)\|^2 \le CM(t)^{2\gamma}\left(\|Au'(t)\|^2+\|A^{1/2}u'(t)\|^2\frac{\|Au(t)\|^2}{M(t)}\right)$$
$$\le C(1+t)^{-4-\frac{1}{\gamma}}$$

where we used the estimates (2.7), (3.2), and (3.3) at the last inequality. Therefore, we conclude the desired estimate (3.15).

Moreover, from Equation (1.1) we observe

$$M(t)^\gamma\|Au(t)\|_{H^1}\le\rho\|u''(t)\|_{H^1}+\|u'(t)\|_{H^1}\,,$$

and from (2.8), (3.3), and (3.4) that

$$\|Au(t)\|_{H^1}^2\le C\left(\|u''(t)\|_{H^1}^2+\|u'(t)\|_{H^1}^2\right)M(t)^{-2\gamma}\le C(1+t)^{-\frac{1}{\gamma}}$$

which implies the desired estimate (3.16). □

Gathering Proposition 3.1, Proposition 3.2, and Proposition 3.3, we arrive at the following theorem.

Theorem 3.4 *In addition to the assumption of Theorem 2.1, suppose that the initial data* $[u_0,u_1]$ *belong to* $\mathcal{D}(A^{3/2})\times\mathcal{D}(A)$*. Then, the solution* $u(t)$ *of* (1.1) *satisfies*

$$C'(1+t)^{-\frac{1}{\gamma}}\le\|A^{k/2}u(t)\|^2\le C(1+t)^{-\frac{1}{\gamma}}\quad\text{for}\quad k=0,1,2,3\,, \tag{3.17}$$
$$\|u'(t)\|_{H^2}^2\le C(1+t)^{-2-\frac{1}{\gamma}}\,, \tag{3.18}$$
$$\|u''(t)\|^2\le C(1+t)^{-4-\frac{1}{\gamma}}\quad\text{for}\quad t\ge 0\,, \tag{3.19}$$

where C *and* C' *are some positive constants.*

Proof. (3.17) follows from (3.1) and (3.16). (3.18) follows from (3.3). (3.19) follows from (3.15). □

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