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Axiomatic Set Theory

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Abstract

In this paper, we study the new axiomatic set theory.

Here we give the new exact definition of the concept of sets and the proof of its existence theorem by using the axiomatic method.

Thus we give the complete solution of the problem of axiomatic set theory.

Thereby we obtain the complete solution of the problem of the foundation of the mathematics.

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Introduction

In this paper, we define the concept of sets and prove its existence theorem by using the axiomatic method. As for this paper, we refer to Ito[3], Chapter 1.

We know Cantor's paradox of the set theory. This means that we cannot consider simply the set of all sets as a set. Therefore, we have to restrict the method of definition of the sets. Therefore we must create the new axiomatic set theory. Thereby, we define new concept of sets by the conditions which should be satisfied by all sets. We say that the necessary and sufficient conditions for the definition of all sets is the complete system of axioms of the set theory.

Until now, as the system of axioms of the set theory, we generally use the system of axioms of the set theory ZFC.

This is the combined axioms of Zermelo-Fraenkel's system of axioms ZF and the axiom of choice C. If the set theory satisfies the system of axioms ZF, we say that it is the ZF set theory. If the set theory satisfies the system of axioms ZFC, we say that it is the ZFC set theory.

Here, the definitions of the concepts of sets are the definition of the class of all sets satisfying the system of axioms ZF and also the definition of the class of all sets satisfying the system of axioms ZFC.

It is considered that we can prove the existence theorems of the concepts of the sets by constructing a model of the ZF set theory and a model of the ZFC set theory.

Further, in the axiomatic set theory until now, we do not distinguish clearly the prescription of the system of axioms for the definition of the concept of sets and the construction of a model of the concept of sets.

Recently, in order to prove the existence theorem of the class of all sets satisfying the system of axioms ZFC, we have to add the axiom of the ordinal numbers and the axiom of the transcendental construction.

For that purpose, in this paper, we define the set theory I and prove its existence theorem of the set theory I by using the system of axioms I. Here the system of axioms I is the combined system of the system of axioms ZFC and the axiom of the ordinal numbers and the axiom of the transcendental construction.

Further, in the expression of the system of axioms I, we prescribe the minimum of the necessary relations among the relations for the symbols $\in$, $=$, $\emptyset$, $\cup$, $\cap$, $\subset$ etc. describing the relations and the operations of sets.

Thereby the all relations of all sets of the set theory I are derived deductively.

We prove the existence theorem of the set theory I by constricting a model. This is the proof by using the reduction to absurdity. Thereby we prove the consistency of the system of axioms I.

As for the axiomatic set theory, we refer to Matsumura [5], Takeuti [9] $\sim$ [11] and Iwanami Dictionary of Mathematics, 4th ed., [8].

Especially, as for the system of axioms ZF and the system of axioms ZFC, we refer to Iwanami Dictionary of Mathematics, 4th ed., [8].

At last, we express my heartfelt thanks to my wife Mutuko for her cooperation of making this file of the manuscript.

1 Preparation for the terminology and the notation

In this section, we give the necessary preparation for the classical logic and the notation used in this paper.

We assume that the logic symbols

$$\vee,\ \wedge,\ \neg,\ \rightarrow,\ \forall,\ \exists$$

used in the explanation of the system of axioms I of the set theory in the following are defined in the classical logic. Especially, we prove the consistency of the system of axioms I of the set theory I by constructing a model of the set theory I. This is the proof by using the reduction to absurdity. We remark that the method of proof by using the reduction to absurdity is right because the law of excluded middle is right in the classical logic.

In the classical logic used here, $\vee$ denotes the logical sum, $\wedge$ denotes the logical product, $\neg$ denotes the negation and $\rightarrow$ denotes the implication. $\forall$ and $\exists$ denote the quantifiers.

Now we assume that A and B are two formulas. Then $A \vee B$ means "A or B", $A \wedge B$ means "A and B", $\neg A$ means "not A" and $A \rightarrow B$ means "if A holds, then B holds". $\forall x A$ is said to be a universal formula and $\exists x B$ is said to be an existential formula.

$\forall x A$ means "for all x, A holds", $\exists x B$ means "for a certain x, B holds".

$A \leftrightarrow B$ is defined as $(A \rightarrow B) \wedge (B \rightarrow A)$.

Here we explain the notation used afterward.

We assume that a set considered in this section is a set in the set theory I. Therefore, all letters appeared in the statements about the system of axioms I denote the sets. The symbol $x \in y$ means that x is an element of y.

We assume that $A(a)$ denotes the proposition satisfied for a set a.

Then we denote the class of all x satisfying $A(x)$ as $\{x;\ A(x)\}$.

Especially, if $A(x)$ denotes $x \notin x$, this means $\{x;\ x \notin x\}$. $\{x;\ x \notin x\}$ cannot be a set by virtue of Russell's paradox. Therefore, in general, we say that $\{x;\ A(x)\}$ is a **class**.

Definition 1.1 Assume that a class A is $A = \{x;\ A(x)\}$.

Then we define that A is a set if we have

$$a(a = A).$$

Here a denotes a certain set.

Therefore, in general, we define that A is a class and not a set.

In the sequel, we say simply that a class is a **set** if the class is a set. Against this, we say simply that a true class is a **class**.

Then we have the proposition in the following.

Proposition 1.1 *We use the notation in Definition* 1.1. *Then the statements* (1) $\sim$ (3) *in the following are equivalent*:

(1) $a = A$.

(2) $\forall x(x \in a \leftrightarrow x \in A)$.

(3) $\forall x(x \in a \leftrightarrow A(x))$.

2 Definition of the concept of sets

In this section, we give the complete system of axioms I of the set theory for the purpose of prescribing the concept of sets. The system of axioms I of the set theory is the system of axioms which is the combined system of the system of axioms ZFC and the axiom of the ordinal numbers and the axiom of the transcendental construction.

Namely, the system of axioms I is the combined system of the axiom of extensionality, the axiom of empty set, the axiom of unordered pair, the axiom of union, the axiom of power set, the axiom of subset, the axiom of infinity, the axiom of ordinal numbers, the axiom of image set, the axiom of regularity, the axiom of choice and the axiom of transcendental construction.

We say that the complete system of axioms I prescribing the set theory I is the system of accurate conditions which are satisfied by all sets appeared in the set theory I. The system of axioms I is the complete system of axioms of the set theory I.

In this section, we formulate the new definition of the concept of sets. In the definition of the concept of sets, the symbol $\in$ is given at the beginning. Namely, for two sets a and A, it is determined that $a \in A$ is true or false.

Further the symbols $=$, $\emptyset$, $\cup$, $\cap$, $\subset$ etc. of the relations of sets and operations are given beforehand.

Remark 2.1 Historically, the definitions of the symbols $=$, $\emptyset$, $\cup$, $\cap$, $\subset$ are defined concretely in the system of axioms ZF or in the system of axioms ZFC. Really, when we prescribe the axioms, these symbols are prescribed in the beginning and it is important that we prescribe the conditions satisfied by these symbols and operations.

Then, we prescribe the accurate relations among the relations satisfied by the relations or the operations of sets $=$, $\emptyset$, $\cup$, $\cap$, $\subset$ etc., in the expression of the system of axioms I and it is important that the all relations of sets in the set theory I are derived inductively from these accurate relations. If the set theory satisfies the system of axioms I, we say that this set theory is the set theory I.

We give the definition of the concept of sets in the set theory I in the following Definition 2.1.

Definition 2.1 (Definition of the concept of sets) We define that the class V is the class of all sets if the axioms (I) $\sim$ (XI) in the following are satisfied.

(I) In V, the relation $\in$ is defined. Namely, for arbitrary elements a and A in V, it is determined that $a \in A$ is true or false. We denote $a \notin A$ if $a \in A$ does not hold. Then $A \in V$ means that A is an element of V and we say that A is a **set**. If A is a set, $a \in A$ means that a is an **element** of A.

In the sequel, the letter other than V denotes a set. Then we have (I-1) $\sim$ (I-6) in the following.

(I-1) We have $V \notin V$. Namely V is not a set and V is a true class.

(I-2) If we have $A \in V$, we have $A = \{a \in V;\ a \in A\}$.

(I-3) For two elements A and B in V, $A = B$ means that we have the condition

$$(x \in A) \leftrightarrow (x \in B).$$

(I-4) For arbitrary elements A, B and C in V, we have (i) $\sim$ (iii) in the following:

(i) We have $A = A$. Therefore we have $A \notin A$.

(ii) If we have $A = B$, we have $B = A$.

(iii) If we have $A = B$ and $B = C$, we have $A = C$.

(I-5) For three elements A, B and C in V, we have $A \in C$ if we have $A \in B$ and $B \in C$.

(I-6) There exists $\emptyset \in V$ such that, for an arbitrary element a in V, we have $a \notin \emptyset$. We say that $\emptyset$ is the **empty set**.

(II) For two sets a and b, there exists a set $\{a,\ b\}$. Especially, we denote $\{a,\ a\}$ as $\{a\}$.

(III) For two elements A and B in V, three operations $A \cup B$, $A \cap B$ and $A - B$ are defined. We say that $A \cup B$ is a **union** of A and B and $A \cap B$ is a **product set** of A and B. Further we also say that $A \cap B$ is an **intersection** of A and B. We say that $A - B$ is a **difference set** which is a subtraction of B from A. Especially, if we have $B \subset A$, we say that $A - B$ is a **complementary set** of B for A.

Then we have the statements (III-1) $\sim$ (III-5) in the following:

(III-1) For three elements A, B and C in V, we have the following (i) $\sim$ (iv):

(i) $A \cup B = B \cup A$.

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$.

(iii) $A \cup A = A$.

(iv) $A \cup \emptyset = A$.

(III-2) For three elements A, B and C in V, we have the following (i) $\sim$ (iv):

(i) $A \cap B = B \cap A$.

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$.

(iii) $A \cap A = A$.

(iv) $A \cap \emptyset = \emptyset$.

(III-3) For three elements A, B and C in V, we have the following (i) and (ii):

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(III-4) For three elements A, B and C in V, we have the following (i) and (ii):

(i) $A \cup (A \cap B) = A$.

(ii) $A \cap (A \cup B) = A$.

(III-5) For three elements A, B and C in V, we have the following (i) and (ii):

(i) $A - B = \{x \in A;\ x \notin B\}$.

(ii) $A - (B \cup C) = (A - B) \cap (A - C)$.

(IV) If, for two elements A and B in V, the relation $A \subset B$ is defined, we say that A is s **subset** of B. Then we have the following (i) $\sim$ (v):

(i) We assume that we have a set A and a proposition $P(x)$ defined for all elements x of A, and it is determined that $P(x)$ is true or false.

Then we assume that $B = \{x \in A;\ P(x)\}$ is the set of all elements x in A such that $P(x)$ is true. Then we have $B \subset A$.

(ii) We have $A \subset A$. Therefore, $A \subset A$ includes the equality $A = A$. If we have $A \subset B$ and $A \neq B$, we denote this as $A \subsetneqq B$.

(iii) If we have $A \subset B$ and $B \subset A$, we have $A = B$.

(iv) If we have $A \subset B$ and $B \subset C$, we have $A \subset C$.

(v) If we have $\emptyset \in V$, we have $\emptyset \subset A$ for an arbitrary $A \in V$.

(V) For an arbitrary element A in V, there exists a set $P(A)$ of all subsets of A. Then we have

$$P(A) = \{B;\ B \subset A\}.$$

We say that the set $P(A)$ is the **power set** of the set A.

(VI) There exists A in V such that we have the following (i) $\sim$ (iii):

(i) We have $\emptyset \in A$.

(ii) For $x \in A$, we have $x' = x \cup \{x\} \in A$.

(iii) We have $A = \{\emptyset\} \cup \{x';\ x' = x \cup \{x\},\ (x \in A)\}$.

Then the set A is an infinite set.

(VII) There exists the subclass O_n of V such that we have the following (i) $\sim$ (iv):

(i) For α and β in O_n, we define that $\beta < \alpha$ holds if we have $\beta \in \alpha$.

(ii) There exists only one empty set $\emptyset$ in O_n.

(iii) If $\alpha \in O_n$ is not the empty set, $\alpha = \{\beta;\ \beta < \alpha\}$ satisfies either one of the following conditions (iii-1) or (iii-2):

(iii-1) If α includes the maximum element β, we have

$$\alpha = \beta \cup \{\beta\}.$$

(iii-2) If α does not include the maximum element, we have

$$\alpha = \{\beta;\ \beta < \alpha\} = \bigcup_{\beta<\alpha} \{\beta\}.$$

(iv) We have $O_n = \{\emptyset\} \cup \{\alpha;\ \alpha = \{\beta \in O_n;\ \beta < \alpha\}\}$.

We define that an element of O_n is an **ordinal number**.

The set A in (VI) in the above is the set which belongs to O_n. This set A is the set of all finite ordinal numbers.

(VIII) For a set A, there exists a set B such that, for an arbitrary element $x \in A$, we can correspond $y \in B$ uniquely so that we have $y = f(x)$.

Namely, a mapping $f:\ A \to B$ is defined and we have

$$B = f(A) = \{y;\ y = f(x),\ (x \in A)\}.$$

(IX) If a is a nonempty set, there exists an element b in a so that a and b does not include any common element. Namely, we have

$$\forall a \left[\ (a \neq \emptyset) \to \exists b\{(b \in a) \wedge (a \cap b = \emptyset)\}\ \right].$$

By virtue of (IX), we see that an arbitrary set a satisfies the conditions $a \notin a$.

(X) If X is a nonempty set, there exists a mapping f of $P(X) - \{\emptyset\}$ into X such that it satisfies the following condition (i):

(i) If we have $A \in P(X) - \{\emptyset\}$, we have $f(A) \in A$.

Here we say that the mapping f is the **choice function**.

(XI) For an arbitrary ordinal number α in O_n, there corresponds a set V_α such that we have the following conditions (i) $\sim$ (iii):

(i) For the ordinal number 0, V_0 is the empty set $\emptyset$.

(ii) If an ordinal number α is not 0, we have either one of the following conditions (ii-1) or (ii-2):

(ii-1) If we have $\alpha = \beta \cup \{\beta\}$, we have

$$V_\alpha = P(V_\beta).$$

(ii-2) If $\alpha = \{\beta;\ \beta < \alpha\}$ does not include the maximum element, we have

$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta.$$

(iii) Then we have

$$V = \bigcup_{\alpha} V_\alpha.$$

3 Definition of the ordinal numbers and its existence theorem

In this section, we study the definition of the ordinal numbers and its existence theorem. As for the details, we refer to Matsumura [5], Takeuti [11], and Japanese Society of Mathematics [8].

Here we define the ordinal numbers.

Definition 3.1 We define that the class O_n of all sets which satisfy the condition (VII) in Definition 2.1 is the **class of ordinal numbers**. We define that an element of O_n is an **ordinal number**.

Especially, if α satisfies the condition (iii-1) of (VII) in Definition 2.1, we say that this α is an **isolated ordinal number**. We denote this as $\alpha = \beta + 1$. If α satisfies the condition (iii-2) of (VII) in Definition 2.1, we say that α is a **limit ordinal number**.

Corollary 3.1 *If we have $\beta < \alpha$ for $\alpha,\ \beta \in O_n$, we have $\beta \subset \alpha$.*

Theorem 3.1 *We define* $\alpha \leq \beta$ *for* $\alpha,\ \beta \in O_n$ *if we have* $\alpha = \beta$ *or* $\alpha < \beta$. *Then, for* $\alpha,\ \beta$ *and* $\gamma \in O_n$, *we have the following* (1) $\sim$ (4):

(1) *For* α *and* β, *we have only one formula among the following* (i) $\sim$ (iii):

(i) $\alpha < \beta$. (ii) $\alpha = \beta$. (iii) $\alpha > \beta$.

(2) *We have* $\alpha \leq \alpha$.

(3) *If we have* $\alpha \leq \beta$ *and* $\beta \leq \alpha$, *we have* $\alpha = \beta$.

(4) *If we have* $\alpha \leq \beta$ *and* $\beta \leq \gamma$, *we have* $\alpha \leq \gamma$.

By virtue of Theorem 3.1, we see that O_n is the **totally ordered set**.

We happen to say that the totally ordered set is the **linearly ordered set**.

Here we propose **Peano's system of axioms**. Peano's system of axioms gives the definition of the finite ordinal numbers.

Definition 3.2 (Peano's system of axioms) We define that the set $\boldsymbol{N}$ of some sets is the **set of finite ordinal numbers** if we have the following conditions (i) $\sim$ (iii):

(i) We have $\emptyset \in \boldsymbol{N}$.

(ii) If we have $\alpha \in \boldsymbol{N}$, we have $\alpha' = \alpha \cup \{\alpha\} \in \boldsymbol{N}$.

(iii) We have $\boldsymbol{N} = \{\emptyset\} \cup \{\alpha'; \alpha' = \alpha \cup \{\alpha\},\ (\alpha \in \boldsymbol{N})\}$
(**Axioms of mathematical induction**).

We define that an element of $\boldsymbol{N}$ is a **finite ordinal number**.

The conditions of the definition 3.1 of the ordinal numbers mean that we replace the axiom of mathematical induction in the conditions of the Definition 3.2 of the finite ordinal numbers with the axiom of transcendental induction.

By using the axiom of transcendental induction, we can prove that the proposition $A(\alpha)$ concerning an ordinal number α holds for an arbitrary ordinal number $\alpha \in O_n$. In this way, we say that the method of proof by using the axiom of transcendental induction is the **principle of transcendental induction**. As the form of transcendental induction usually used, we have the theorem in the following.

Theorem 3.2 (Method of transcendental induction) *We use the notation in* Definition 3.1. *We have the formula*

$$\forall\alpha \left(\forall\beta < \alpha A(\beta) \to A(\alpha)) \to \forall\alpha A(\alpha) \right).$$

Then we define a **function** $R(\alpha)$ **by the method of transcendental induction**. This is the method of defining the value of the function one by one according to the order of the ordinal numbers.

We define the function $R(\alpha)$ by the method of transcendental induction in the following:

(A) We define $R(\emptyset) = \emptyset$.

(B) For an ordinal number α, we assume that $R(\beta)$ is defined for all $\beta < \alpha$. Then we have the definitions (B.1), (B.2) in the following:

(B.1) If α is an isolated ordinal number, we have β so that we have $\alpha = \beta + 1$. Then we define

$$R(\alpha) = P(R(\beta)).$$

(B.2) If α is a limit ordinal number, we define

$$R(\alpha) = \bigcup_{\beta<\alpha} R(\beta).$$

(C) We define $R(\alpha)$ for all ordinal number α according to the order of the ordinal numbers.

By virtue of this definition, when we define $R(\alpha)$ one by one starting from the ordinal number $\emptyset$, we complete the definition of the function $R(\alpha)$ for all ordinal numbers by defining the value of $R(\alpha)$ for each ordinal number α according to the processes (A), (B) and (C).

Thereby we can consider that the definition of the function $R(\alpha)$ for all ordinal numbers α is determined uniquely.

For the function $R(\alpha)$ defined in the above, we have Theorem 3.3 $\sim$ Theorem 3.5 in the following.

Theorem 3.3 *We have the three formulas in the following*:

(1) $R(\emptyset) = \emptyset$.

(2) $R(\alpha + 1) = R(\alpha) \cup P(R(\alpha))$.

(3) *If α is a limit ordinal number, we have*

$$R(\alpha) = \bigcup_{\beta<\alpha} R(\beta).$$

Theorem 3.4 *We have the three properties in the following*:

(1) $\alpha < \beta \to R(\alpha) \subset R(\beta) \wedge R(\alpha) \in R(\beta)$.

(2) *The sets* $R(\alpha)$ *are transitive. Namely we have the condition*

$$\forall x,\ y\ \Big(\ x \in y \wedge y \in R(\alpha) \to x \in R(\alpha)\ \Big)\ .$$

(3) $a \in R(\alpha),\ b \subset a \to b \in R(\alpha)$.

Theorem 3.5 *We have two properties in the following*:

(1) $R(\alpha + 1) = P(R(\alpha))$.

(2) *We have*

$$\forall x \exists \alpha (x \in R(\alpha)).$$

Namely we have

$$V = \bigcup_{\alpha \in O_n} R(\alpha).$$

Theorem 3.6 *We have the formula*

$$0 \neq A \subset O_n \to \exists \alpha \in A \forall \beta \in A(\alpha \leq \beta).$$

We say that Theorem 3.6 is the **principle of transcendental induction**.

We denote the minimum limit ordinal number as ω. Then $\alpha \in O_n$ is a **finite ordinal number** if and only if we have the condition $\alpha < \omega$.

Theorem 3.7 *The minimum limit ordinal number* ω *in* O_n *is the set* $\omega = \{\emptyset,\ 1,\ 2,\ \cdots\}$ *of all finite ordinal numbers.*

Next, we prove the existence theorem of the ordinal numbers on the basis of the principle of transcendental induction.

Namely we prove this by **constructing a model of ordinal numbers**.

Theorem 3.8 (the existence theorem of the ordinal numbers) *We construct the class* O_n *whose elements are the sets so that we have the following conditions* (1) $\sim$ (4):

(1) *For* α *and* β *in* O_n, *we define that* $\beta < \alpha$ *holds if* $\beta \in \alpha$ *holds.*

(2) *By constructing the empty set* $\emptyset$, *we define the empty set* $\emptyset$ *as* $0 \in O_n$.

(3) *If* $\alpha \in O_n$ *is not* 0, *we can construct the set* $\{\beta;\ \beta < \alpha\}$ *of all ordinal numbers* $\beta < \alpha$ *so that we have the following* (i) *or* (ii):

(i) *If there exists the maximum element* β *of the set* $\{\beta; \beta < \alpha\}$, *we have*

$$\alpha = \beta + 1 = \beta \bigcup \{\beta\}.$$

(ii) *If there does not exist the maximum element of* α, α *is a limit ordinal number and we have*

$$\alpha = \{\beta;\ \beta < \alpha\} = \bigcup_{\beta < \alpha} \{\beta\}.$$

(4) *We have the formula*

$$O_n = \{0\} \cup \{\alpha;\ \alpha = \{\beta \in O_n;\ \beta < \alpha\}\}.$$

Then O_n *is the class of all ordinal numbers. Therefore an element of* O_n *is an ordinal number.*

We see that O_n constructed in the above satisfies the condition (VII) of Definition 2.1. Hence we can construct a model of ordinal numbers by using the principle of transcendental induction.

Thereby we prove the existence of a model of ordinal numbers. Therefore we have the existence theorem of ordinal numbers in the following.

Theorem 3.9 *We use the notation in* Theorem 3.8. *Then we have a model of the class of all ordinal numbers. Therefore there exists the class* O_n *of all ordinal numbers.*

Next we prove that the set $\boldsymbol{N}$ of all finite ordinal numbers satisfies Peano's system of axioms. Namely we have the following theorem.

Theorem 3.10 *We denote the set* $\boldsymbol{N} = \{\alpha;\ \alpha < \omega\}$ *of all finite ordinal numbers as* $\boldsymbol{N} = \{0,\ 1,\ 2,\ \cdots\}$. *Then we have the following conditions* (1) $\sim$ (3):

(1) *We have* $0 \in \boldsymbol{N}$.

(2) *If we have* $a \in \boldsymbol{N}$, *then we have* $a + 1 \in \boldsymbol{N}$.

(3) *We have the formula* $\boldsymbol{N} = \{0\} \cup \{a + 1 = a \cup \{a\};\ a \in \boldsymbol{N}\}$.

Therefore the set $\boldsymbol{N} = \{0,\ 1,\ 2,\ \cdots\}$ of all finite ordinal numbers satisfies Peano's system of axioms.

In this sense, Peano's system of axioms gives the definition of finite ordinal numbers. By using the set $\boldsymbol{N}$ of all finite ordinal numbers, we can construct a model of natural numbers.

Here we define the operations of the addition and the multiplication and the order relation by using those defined for the finite ordinal numbers.

4 Construction of a model of the set theory

In this section, we prove the existence theorem of the concept of sets. For that purpose, we prove this by constructing really a model of the class V of all sets defined in Definition 2.1.

We construct a model of the set theory I and we prove that this model satisfies the conditions of the definition of the set theory in Definition 2.1. For that purpose, assuming that the fundamental relation $\in$ is given, we define concretely the relations concerning the sets such as $=$, $\emptyset$, $\cup$, $\cap$, $\subset$ etc. as the necessary symbols and we have only to prove that the conditions in Definition 2.1 are satisfied concerning these symbols. For that purpose, we prescribe the conditions (1) $\sim$ (12) in Theorem 4.1. By virtue of these prescriptions, we can prove the formulas in the axioms of Definition 2.1 concerning these symbols $=$, $\emptyset$, $\cup$, $\cap$, $\subset$ etc..

For the proof of the existence theorem of the set theory I, the concept of ordinal numbers and the axioms of transcendental induction play the fundamental role. As for the details, we refer to Takeuti [12] .

Therefore we have Theorem 4.1 in the following.

Theorem 4.1 (Construction of the class of all sets) *A model of the class V of all sets is constructed as the class of all sets satisfying the following conditions* (1) $\sim$ (12):

(1) **(Axiom of extensionality)** *x and y are equal, namely we have $x = y$ if, for an arbitrary z, $z \in x$ and $z \in y$ are equivalent. Namely, we have*

$$(x = y) \leftrightarrow \forall z\big[(z \in x) \leftrightarrow (z \in y)\big].$$

(2) **(Axiom of empty set)** *There exists a set which has not any element. Namely we have*

$$\exists x \forall y (\neg y \in x).$$

The set considered in the condition (2) *is determined uniquely by virtue of the condition* (1).

We denote this set as $\emptyset$ and we say that this is the **empty set**.

(3) **(Axiom of unordered pair)** *For two arbitrary sets a and b, there exists the set whose elements are only a and b. Namely we have*

$$\forall a \forall b \exists p \forall u \; \big[\; (u \in p) \leftrightarrow (u = a) \vee (u = b) \; \big] \; .$$

Definition 4.1 We say that p in the condition (3) is the **unordered pair** of a and b and we denote it as $\{a,\ b\}$. We denote $\{a,\ a\}$ as $\{a\}$.

(4) **(Axiom of union)** *For an arbitrary set a, there exists the union s of all sets of a. Namely we have*

$$\forall a \exists s \forall u \left[(u \in s) \leftrightarrow \exists x \{(u \in x) \wedge (x \in a)\} \right].$$

Definition 4.2 We say that s in the condition (4) is the sum of a and we denote it as $\cup a$. When we have $a = \{A,\ B\}$, we happen to denote $\cup a$ as $A \cup B$.

Definition 4.3 When every element of x also belongs to y surely, we say that x is a subset of y and we denote it as $x \subset y$. Namely we have

$$x \subset y \leftrightarrow \forall u \left[(u \in x) \rightarrow (u \in y) \right].$$

(5) **(Axiom of power set)** *For an arbitrary set a, there exists the set t of all subsets of a. Namely we have the formula*

$$\forall a \exists t \forall u \left[(u \in t) \leftrightarrow (u \subset a) \right].$$

Definition 4.4 We denote the set t in the condition (5) as $P(a)$.

(6) **(Axiom of subset)** *For an arbitrary set a and the accurate proposition $A(x)$ which has the meaning for all elements x of a, there exists the set y of all elements x of a such as $A(x)$ is true for x. Namely we have the formula*

$$\forall a \exists y \forall x \left[(x \in y) \leftrightarrow \{(x \in a) \vee A(x)\} \right].$$

The set y in the condition (6) happen to be expressed as

$$y = \{x \in a;\ A(x)\}.$$

Then we say that y is a **subset** of a. Namely we have $y \subset a$.

By virtue of the condition (6), the set $P(a)$ in Definition 4.4 can be expressed as

$$P(a) = \{u;\ u \subset a\}.$$

Definition 4.5 **(Definition of product set)** We define that the **product set** $A \cap B$ of two sets A and B is the set

$$A \cap B = \{x;\ x \in A \text{ and } x \in B\}.$$

We happen to say that $A \cap B$ is the **intersection** of A and B.

By virtue of the condition in the above, we can prove the existence of the product set.

Thereby we can define the concept of a mapping.

(7) **(Axiom of infinity)** *There exists a set A with the properties* (i) *and* (ii) *in the following*:

(i) *We have $\emptyset \in A$.*

(ii) *For $x \in A$, these exists the successor x^+ of x and we have $x^+ \in A$.*

When we consider the set A in the condition (7) as a model of the natural numbers, we consider the empty set as the number 0.

Next we give the axiom which prescibes the ordinal numbers.

(8) **(Axiom of ordinal numbers)** *There exists the class O_n of sets as its elements so that we have the following conditions* (I) $\sim$ (IV):

(I) *For α and β in O_n, we define that $\beta < \alpha$ holds if $\beta \in \alpha$ holds.*

(II) *These exists only one element in O_n which is called as the empty set $\emptyset$.*

(III) *If $\alpha \in O_n$ is not the empty set $\emptyset$, $\alpha = \{\beta;\ \beta < \alpha\}$ satisfies either one of the conditions* (i) *or* (ii) *in the following*:

(i) *If α contains the maximum β, we have*

$$\alpha = \beta \cup \{\beta\}.$$

(ii) *If α does not contain the maximum, we have*

$$\alpha = \{\beta;\ \beta < \alpha\} = \bigcup_{\beta<\alpha} \{\beta\}.$$

Here the symbol $\beta < \alpha$ means that we have the condition $\beta \in \alpha$ for two sets α and β in O_n.

(IV) *We have the formula*

$$\forall\alpha \left(\forall\beta < \alpha\ \beta \in O_n \to \alpha \in O_n \right) \to \forall\alpha\ \alpha \in O_n.$$

The (IV) in the condition (8) is equivalent to (IV)$'$ in the following:

(IV)$'$ *We have the formula*

$$O_n = \{\emptyset\} \cup \{\alpha;\ \alpha = \{\beta \in O_n;\ \beta < \alpha\}\}.$$

We say that (IV) or (IV)$'$ in the above is the **axiom of transcendental induction**.

(9) **(Axiom of Image)** *Assume that we have a proposition $F(x,\ y)$ including two free variables and one set A, and assume that, for every element x of A, there exists at most one element y such that $F(x,\ y)$*

holds. Then there exists a set B of all elements y so that, for an element $x \in A$, $F(x,\ y)$ holds. Namely, we have the formula

$$\forall x \left[(x \in A) \to \forall u \forall v \{(F(x,\ u) \wedge F(x,\ v)) \to (u = v)\} \right]$$

$$\to \exists \beta \forall t \left[(t \in \beta) \leftrightarrow \exists x \{(x \in A) \wedge F(x,\ t)\} \right].$$

By virtue of the condition (9), for an arbitrary $x \in A$, we can correspond uniquely an element $y \in B$ so that $y = f(x)$ holds and the mapping $f:\ A \to B$ is defined.

Then we say that B is the image of A by the mapping f. We happen to say simply that B is the **image** of A by virtue of f. Then we denote this as

$$B = f(A).$$

(10) **(Axiom of regularity)** *If a is not the empty set, there exists an element b of a so that a and b does not include any common element. Namely we have the formula*

$$\forall a \ [\ (a \neq \emptyset) \to \exists b \{(b \in a) \wedge (a \cap b = \emptyset)\}\ ].$$

By using the condition (10), we can conclude that every set does not include itself as an element. Namely **every set *a* satisfies the condition** $a \notin a$.

(11) **(Axiom of choice)** *If X is not the empty set, there exists a mapping f of $P(X) - \{\emptyset\}$ into X such that we have the condition* (1) *in the following*:

(1) *If we have*

$$A \in P(X) - \{0\},$$

we have

$$f(A) \in A.$$

Here we say that the mapping f is the **choice function**.

Next, we give the axiom of transcendental construction concerning the existence of the class V of all sets.

(12) **(Axiom of transcendental construction)** *The set V_α corresponds to every ordinal number α of O_n such that we have the following* (a) $\sim$ (c):

(a) *For the ordinal number* 0, *V_0 is the empty set $\emptyset$.*

(b) *For an ordinary number α which is not* 0, *V_α satisfies either one of the following* (i) *or* (ii):

(i) *If* $\alpha = \beta + 1$ *holds, we have*

$$V_\alpha = P(V_\beta).$$

(ii) *If* α *is a limit ordinal number, we have*

$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta.$$

(c) *For the class* V *of all sets which satisfy the conditions* (1) $\sim$ (11), *we have*

$$V = \bigcup_{\alpha} V_\alpha.$$

We happen to say that the condition (12) of Theorem 4.1 is the **Axiom of constructibility**. Thereby we show the construction of a model of the sets of the set theory I.

Therefore V is a model of the set theory I. Thereby the consistency of the system of axioms I is proved since the existence of a model V of the set theory I is proved. Here we have the existence theorem of the set theory I in the following.

Theorem 4.2 (Existence theorem) *The system of axioms* I *is consistent. Therefore there exists the set theory* I.

Remark 4.1 In the set theory until now, the "ordinal number" is the "order type of the well-ordered set".

As the premise, we need all sets and the classification of the ordered set by using the order type.

Then it is troubled for the consideration so that we proceed the construction of a model of the axiomatic set theory along the generation of the ordinal numbers. Before we prove the existence of a model of the set theory, we must prove the existence of all sets.

By virtue of my method, the "definition of the ordinal numbers and its existence theorem" is completed in itself. It is no problem to construct a model of the set theory by using it. In this sense, my axiomatic method is fatally important for the solution of the problem of the foundation of mathematics.

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