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Length of Integer Solutions of Linear Diophantine Equations and Related Problems

By

Shin-ichi Katayama

Professor Emeritus of Tokushima University,
e-mail address : *shinkatayama@tokushima-u.ac.jp*

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Abstract

We shall introduce a length of the integer solutions of linear diophantine equations and investigate the fundamental properties of this length. We will also give an application of this length to a famous mathematical puzzle called *three jug problem*.

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1 Introduction

Probably the simplest diophantine equation may be the following linear diophantine equation of two valuables x, y,

$$E_{(a,b;c)} : ax + by = c, \text{ where } a, b, c \in \mathbb{Z}.$$

We shall denote the integer solutions of $E_{(a,b;c)}$ as $S_{(a,b;c)}$. It is well known the above equation has the integer solutions (x, y) if and only if $\mathrm{GCD}(a, b)|c$ and all the solutions are explicitly obtained by using the Euclidean Algorithm.

Let us start an example $E_{(5,3;38)} : 5x + 3y = 38$. We shall explain the usual way of writing down the integer solutions of this equation. Firstly, from the Euclidean Algorithm, one can find the special integer solutions $(x, y) = (-1, 2)$ of the equation $E_{(5,3;1)}$. Multiplying both sides of the equation $E_{(5,3;1)}$ by 38, one obtains the solutions $(x, y) = (-38, 78)$ of the equation $E_{(5,3;38)}$. Then all the integer solutions $S_{(5,3;38)}$ of the equation $E_{(5,3,38)}$ are written as follows;

$$S_{(5,3;38)} = \{(x, y) \mid x = -38 + 3k, y = 78 - 5k, \text{where } k \in \mathbb{Z}\}.$$

We note that this set of integer solutions $S_{(5,3;38)}$ is a residue class of $\mathbb{Z}^2$ modulo $\{k(3, -5) \mid k \in \mathbb{Z}\}$, where $\{k(3, -5) \mid k \in \mathbb{Z}\} \cong \mathbb{Z}$. Therefore $(-38, 76)$ is a representative of the residue class $S_{(5,3;38)}$. But, taking $k = 15$, we can choose another "small" representative $(x, y) = (7, 1)$. In the next section, we shall introduce the length of integer solutions and $(7, 1)$ are really the smallest integer solutions and suitable for the representative of this residue class (see Theorem 2.6 and Remark 2.7).

2 The length of integer solutions

Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n), \boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ be two vectors in $\mathbb{R}^n$. Then the following $d(\mathbf{a}, \mathbf{b})$ defines a different way of measuring the distance of $\mathbb{R}^n$ which is called the Manhattan distance,

$$d(\boldsymbol{a}, \boldsymbol{b}) = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|.$$

Let us denote $(0, 0, \ldots, 0) \in \mathbb{R}^n$ by $\mathbf{0}$. Now treat the linear diophantine equation

$$E_{(a_1,a_2,\ldots,a_n;c)} : a_1x_1 + a_2x_2 + \cdots + a_nx_n = c, \text{where } a_1, a_2, \ldots, a_n, c \in \mathbb{Z}.$$

We shall define the length $L(\boldsymbol{x})$ of the integer solution $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ of the above linear diophantine equation $E_{(a_1,a_2,\ldots,a_n;c)}$ by putting

$$L(x_1, x_2, \ldots, x_n) = d(\boldsymbol{x}, \mathbf{0}) = |x_1| + |x_2| + \cdots + |x_n|.$$

Remark 2.1 *The Manhattan distance for the case $n = 2$ is named after the grid pattern of the streets and avenues in Manhattan.*

In the following, we shall restrict ourselves to the simplest case $n = 2$, i.e.,

$$E = E_{(a,b;c)} : ax + by = c.$$

Then there exist the solutions $(x, y) \in S_{(a,b;c)}$ with the length $L(x, y) = |x|+|y|$ of the minimal value. We denote this minimal value $\min\{L(x, y) \mid (x, y) \in S_{(a,b;c)}\}$ by L_E, and call the value L_E the *minimal length* of the integer solutions of the linear diophantine equation $E = E_{(a,b;c)}$. We also call the solutions (x, y) with the minmal length L_E the *minimal integer solutions*.

Firstly, we shall begin the distribution of the length of integer solutions of an example of the equation $E_{(5,3;38)}$.

Tabel 1

k	$(x = -38 + 3k, y = 76 - 5k)$	The length $L(x, y)$
$\vdots$	$\vdots$	$\vdots$
0	$(-38, 76)$	114
$\vdots$	$\vdots$	$\vdots$
k	$(-38 + 3k, 76 - 5k)$	$114 - 8k$
$\vdots$	$\vdots$	$\vdots$
11	$(-5, 21)$	26
12	$(-2, 16)$	18
13	$(1, 11)$	12
14	$(4, 6)$	10
15	$(7, 1)$	8
16	$(10, -4)$	14
17	$(13, -9)$	22
18	$(16, -14)$	30
$\vdots$	$\vdots$	$\vdots$
k	$(-38 + 3k, 76 - 5k)$	$8k - 114$
$\vdots$	$\vdots$	$\vdots$

Then the above length $L(x, y)$ is classified into the following three arithmetic progressions, which will be abbreviated to AP in the following;

$\{18 + 8k | k \geq 0\}$ AP with the initial term 18 and the common difference 8,
$\{8, 10, 12\}$, Finite AP with the common difference 2,
$\{14 + 8k | k \geq 0\}$ AP with the initial term 14 and the common difference 8.

Now we will generalize the above results to the equation $E_{(a,b;c)} : ax + by$, where $a > b > 0$ and $\text{GCD}(a, b) = 1$ and $c > 0$. Then the length $L(x, y)$ of the integer solutions $S_{(a,b;c)}$ is classified into the following three classes:

Infinite AP with the common difference $a + b$, for $(x, y) \in S_2 = \{(x, y) | x < 0\}$,
Finite AP with the common difference $a - b$, for $(x, y) \in S_0 = \{(x, y) | x, y \geq 0\}$,
Infinite AP with the common difference $a + b$, for $(x, y) \in S_1 = \{(x, y) | y < 0\}$.

Let L_i be the minimal length of the minimal integer solutions in S_i, $(0 \leq i \leq 2)$. We note the case $S_0 = \emptyset$ may happen. For example, any $(x, y) \in S_{(a,b;1)}$ with $a > b \geq 2$ satify $xy < 0$ and hence $S_0 = \emptyset$ for this case.

Theorem 2.2 *Assume* $a > b > 0$, $\mathrm{GCD}(a,b) = 1$ *and* $c > 0$. *Then the minimal length* $L_E = \min(L_0, L_1, L_2)$. *In case* $S_0 = \emptyset$, $L_E = \min(L_1, L_2)$. *In case* $S_0 \neq \emptyset$, $L_E = \min(L_0, L_1)$.

2.1 Algorithm for finding the minimal solutions 1

We shall recall the Euclidean algorithm for $a > b > 0$ with n steps;

$$\begin{array}{lcl} a & = & a_0 b + r_1,\ (0 < r_1 < b), \\ b & = & a_1 r_1 + r_2,\ (0 < r_2 < r_1), \\ r_1 & = & a_2 r_2 + r_3,\ (0 < r_3 < r_2), \\ & \vdots & \\ r_{n-2} & = & a_{n-1} r_{n-1} + r_n,\ (0 < r_n < r_{n-1}), \\ r_{n-1} & = & a_n r_n, \\ r_n & = & d = \mathrm{GCD}(a,b). \end{array}$$

Put $r_{-1} = a$, and $r_0 = b$. Then the binary recurrence sequences X_i, Y_i are defined by putting

$$X_i = a_{i-1} X_{i-1} + X_{i-2}, \quad Y_i = a_{i-1} Y_{i-1} + Y_{i-2},$$

with initial terms $X_{-1} = 1, X_0 = 0$ and $Y_{-1} = 0, Y_0 = 1$, One obtains, by induction,

$$a(-1)^{i-1} X_i + b(-1)^i Y_i = r_i, (-1 \le i \le n).$$

Assume $n \geq 2$, i.e., $b \not| a$. d denotes $\mathrm{GCD}(a,b)$. Then, from the extended Euclidean algorithm, $E_{(a,b;d)}$ has the minimal integer solutions

$$(x,y) = ((-1)^{n-1} X_n, (-1)^n Y_n).$$

Then $(-1)^n X_{n+1} = (-1)^n b, (-1)^{n+1} Y_{n+1} = (-1)^{n+1} a$, and hence $X_i \leq \dfrac{b}{2d}$ and $Y_i \leq \dfrac{a}{2d}$. Therefore the minimal length L_E satisfies

$$L_E = X_n + Y_n < \frac{a+b}{2d}.$$

Since $X_i + Y_i < X_n + Y_n$ for any $-1 \leq i \leq n-1$, one can generalize this result as follows.

Theorem 2.3 *For the case* $c = r_i$ $(-1 \leq i \leq n)$, *the minimal integer solutions of the equation* $E = E_{(a,b;r_i)} : ax + by = r_i$ *and the minimal length* L_E *are given by*

$$(x,y) = ((-1)^{i-1} X_i, (-1)^i Y_i), \quad L_E = X_i + Y_i, (-1 \leq i \leq n).$$

2.2 Continued fraction

Let $\frac{a}{b}$ be a reduced fraction satisfying the following n steps.

$$\begin{array}{lcl} a & = & a_0 b + r_1,\ (0 < r_1 < b) \\ b & = & a_1 r_1 + r_2,\ (0 < r_2 < r_1) \\ r_1 & = & a_2 r_2 + r_3,\ (0 < r_3 < r_2) \\ & \vdots & \\ r_{n-2} & = & a_{n-1} r_{n-1} + r_n,\ (0 < r_n < r_{n-1}) \\ r_{n-1} & = & a_n r_n \\ r_n & = & 1 = GCD(a, b) \end{array}$$

Then the continued fraction expansion of the rational number $\frac{a}{b}$ is denoted by

$$\frac{a}{b} = [a_0; a_1, a_2 \ldots, a_n].$$

The $k-$th intermediate continued fraction is defined by putting

$$\frac{P_k}{Q_k} = [a_0; a_1, \ldots, a_k].$$

Put $P_{-1} = 0, P_0 = 1, Q_{-1} = 1, Q_0 = 0$. The recurrence sequences P_k, Q_k are defied by putting

$$P_{k+1} = a_k P_k + P_{k-1}, Q_{k+1} = a_k Q_k + Q_{k-1}, \text{ for } k \geq 0.$$

These recurrence sequences can be written by the use of matrices,

$$\begin{pmatrix} P_{k+1} & P_k \\ Q_{k+1} & Q_k \end{pmatrix} = \begin{pmatrix} P_k & P_{k-1} \\ Q_k & Q_{k-1} \end{pmatrix} \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $a = P_{n+1}, Q_{n+1} = b$, one gets,

$$\begin{pmatrix} a & P_n \\ b & Q_n \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

We note that $Q_k = X_k,\ P_k = Y_k$, where X_k, Y_k are those of the extended Euclidean algorithm for $a > b$.

2.3 The Frobenius coin problem

To investigate the algorithm of finding the minimal integer solutions for larger c, we shall recall the Frobenius coin problem, which states the existence of the non-negative integer solutions for given linear diophantine equations. Let $\{a_1, a_2, \ldots, a_n\}$ be coprime positive integers. Fron Schur's theorem, there exists the largest positive integer $c = g(a_1, a_2, \ldots, a_n)$ for which the linear diophantine equation $E_{(a_1,a_2,,\ldots,a_n;c)} : a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$ has no non-negative integer solutions $x_1, x_2, \ldots, x_n$. The number $g(a_1, a_2, \ldots, a_n)$ is called the *Frobenius number*. From the definition of the Frobenius number, the equation $E_{(a_1,a_2,\ldots,a_n;c)}$ has non-negative integer solutions for any $c > g(a_1, a_2, \ldots, a_n)$. Though the explicit closed form of the Frobenius number for $n \geq 3$ is still an open problem, the following case $n = 2$ is well known.

Theorem 2.4 (Frobenius number for the case $n = 2$) *Let a, b be coprime positive integers. Then the Frobenius number $g(a, b)$ for the equation $ax+by = c$ is*

$$g(a, b) = ab - a - b.$$

When c varies $0 \leq c \leq g(a, b) = (a - 1)(b - 1) - 1$, there exist exactly $\frac{(a-1)(b-1)}{2}$ equations $E_{(a,b;c)}$ with non-negative integer solutions x, y.

Remark 2.5 *Last part of the above theorem is easily proved from the following property;*

$$E_{(a,b;c)} \text{ has non-negative integer solutions}$$
$$\iff E_{(a,b;g(a,b)-c)} \text{ has no non-negative integer solutions.}$$

2.4 Algorithm for finding minimal solutions 2

Assume the positive integers a, b satisfy $a > b > 0$ and $\mathrm{GCD}(a, b) = 1$. Then the condition for the equation $E_{(a,b;c)} : ax + by = c$ has non-negative integer solutions (x, y) is the following. Consider the following linear congruence $by \equiv c \pmod{a}$. Then there exists $y = y_0$ with $0 \leq y_0 < a$. If $c - by_0 \geq 0$, the integer $x_0 = (c - by_0)/a$ satisfies $ax_0 + by_0 = c$ with $x_0, y_0 \geq 0$, i.e., the equation has the non-negative integer solutions (x_0, y_0). Moreover (x_0, y_0) are the minimal integer solutions of S_0 and hence the length $L(x_0, y_0) = x_0 + y_0$ is L_0. Then the solutions $(x_0 + b, y_0 - a)$ are the minimal solutions of S_1 and the length $L(x_0 + b, y_0 - a) = x_0 - y_0 + a + b$ is the minimal length L_1. Thus

$$L_0 \leq L_1 \iff y_0 \leq \frac{a+b}{2}.$$

On the contrary, if $c - by_0 < 0$, the integer $x_0 = (c - by_0)/a$ satisfy $ax_0 + by_0 = c$ with $x_0 < 0, y_0 \geq 0$, and the equation does not have non-negative integer solutions. The length $L(x_0, y_0) = -x_0 + y_0$ is L_2 for this case. Hence the solutions $(x_0 + b, y_0 - a)$ are the minimal solutions of S_1 and the length $L(x_0 + b, y_0 - a) = x_0 - y_0 + a + b$ is the minimal length L_1. Thus we have

$$L_2 \leq L_1 \iff -x_0 + y_0 \leq \frac{a+b}{2}.$$

Theorem 2.6 *Under the above notations, the minimal integer solutions and the minimal length L_E are the following:*
If $c \geq by_0$ and $y_0 \leq \dfrac{a+b}{2}$, then the minimal integer solutions are (x_0, y_0) and the minimal length is $L_E = x_0 + y_0$.
If $c \geq by_0$ and $y_0 > \dfrac{a+b}{2}$, then the minimal integer solutions are (x_0+b, y_0-a) and the minimal length is $L_E = x_0 - y_0 + a - b$.
If $c < by_0$ and $-x_0+y_0 \leq \dfrac{a+b}{2}$, then the minimal integer solutions are (x_0, y_0) and the minimal length is $L_E = -x_0 + y_0$.
If $c < by_0$ and $-x_0 + y_0 > \dfrac{a+b}{2}$, then the minimal integer solutions are $(x_0 + b, y_0 - a)$ and the minimal length is $L_E = x_0 - y_0 + a + b$.

Remark 2.7 *Given a equation $E_{(a,b;c)} : ax+by = c$, with $a > b > 0$, $c > 0$ and* $\mathrm{GCD}(a, b) = 1$. *From this theorem, one can find the minimal integer solutions (X, Y) and any integer solutions are written in the form $(X+kb, Y-ka), k \in \mathbb{Z}$. When $a = 5, b = 3$ and $c = 37$, one can verifies that $(7, 1)$ are the minimal integer solutions as mentioned in the firsst section.*

3 Related problems

3.1 Equivalence classes of the linear diophantine equation

Let V be the set of integer vectors $(a, b, c) \in \mathbb{Z}^3$ which satisfy the condition $\mathrm{GCD}(a, b)|c$. Then $(a, b, c) \in V$ is nothing but the integer solutions $S_{(a,b;c)} \neq \emptyset$ of the corresponding diophantine equation $E_{(a,b;c)}$. We denote $(a_1, b_1, c_1) \cong (a_2, b_2, c_2)$ if there exists integers $p, q, pq \neq 0$ which satisfy

$$p(a_1, b_1, c_1) = q(a_2, b_2, c_2).$$

Then one can see

$$(a_1, b_1, c_1) \cong (a_2, b_2, c_2) \Longleftrightarrow S_{(a_1,b_1;c_1)} = S_{(a_2,b_2;c_2)}.$$

Therefore, for any $(a, b, c) \in V$, there exists $(a_0, b_0, c_0) \cong (a, b, c)$ with $\mathrm{GCD}(a_0, b_0) = 1$.

Let $\varepsilon_a, \varepsilon_b, \varepsilon_c \in \{-1, 1\}$ and put $a' = \varepsilon_a a, b' = \varepsilon_b b, c' = \varepsilon_c c$. The map ϕ from $E_{(a,b;c)}$ to $E_{(a',b';c')}$ is defined by putting

$$\phi : (x, y) \in S_{(a,b;c)} \to (x', y') \in S_{(a',b';c')},$$

where $x' = \phi(x) = \varepsilon_c \varepsilon_a x, y' = \phi(y) = \varepsilon_c \varepsilon_b y$. Then ϕ defines a bijection from $S_{(a,b;c)}$ to $S_{(a',b';c')}$ which preserves the length of integer solutions

$$L(x, y) = |x| + |y| = |x'| + |y'| = L(x', y') = L(\phi(x), \phi(y)),$$

where

$$ax + by = c \Longleftrightarrow (a\varepsilon_a)(\varepsilon_c\varepsilon_a x) + (b\varepsilon_b)(\varepsilon_c\varepsilon_b y) = (c\varepsilon_c) \Leftrightarrow a'x' + b'y' = c''.$$

Thus, to investigate the distributions of the integer solutions of the given equation $E_{(a,b;c)}$, we may restrict ourselves to the case $a > b > 0, c > 0$ and $\mathrm{GCD}(a, b) = 1$, without loss of generality.

Remark 3.1 *We denote $(a, b, c) \sim (a', b', c')$ if there exists the above map $\phi : E_{(a,b;c)} \to E_{a',b',c')}$. There are examples $(a, b, c) \not\sim (a', b', c')$ and $(a, b, c) \not\cong (a', b', c')$, but have the same set of the length of integer solutions. For example, consider the equations $E_{(11,3;7)}$ and $E_{(9,5;7)}$. Then $(11, 3, 7) \not\sim (9, 5, 7)$ and $(11, 3, 7) \not\cong (9, 5, 7)$, but both equations have the same set of the length of integer solutions $\{7, 21, 28, \ldots\}$, i.e., AP with the initial term 7 and the common difference 14.*

3.2 Three Jug Problem

Originally, *three jug problem* is the following mathematical puzzle. Let a, b are positive integers with $a > b$. Given three jugs, the first jug A with a pints, the second jug B with b pints, and the third jug C with $a + b$ pints. Make two jugs A and C with the same amount $(a + b)/2$, by only completely filling up and/or emptying vessels into others. It is known that this problem can be solved by using the solution of the linear diophantine equation $E_{(a,b;(a+b)/2))}$.

This problem is slightly modified and generalized as follows. Given two empty buckets A and B of positive integer capacities a and b, respectively and a well containing an inexhautible supply of water. Moreover $a > b$ and $\mathrm{GCD}(a, b) = 1$. One is asked to obtain a fixed quantity of liquid c using only two initially empty buckets A and B by only completely filling up and/or emptying buckets into others and also utilizing the well. In the film "Die Hard: With a Vengeance" (1995), this problem of the case $a = 5$, $b = 3$ and $c = 4$ has been treated.

We shall explain this example using the symbol $[p, q]$, where p represents the amount of water in the first bucket A with the capacity 5 and q represents the amount of water in the second bucket B with the capacity 3.

(1)

$$[0, 0] \to [5, 0] \to [2, 3] \to [2, 0] \to [0, 2] \to [5, 2] \to [4, 3]$$

(2)

$$[0, 0] \to [0, 3] \to [3, 0] \to [3, 3] \to [5, 1] \to [0, 1] \to [1, 0] \to [1, 3] \to [4, 0]$$

Here (1) is the procedure corresponding the integer solutions $(2, -2)$ of $E_{(5,3;4)}$: $5x+3y = 4$, and (2) is the procedure corresponding the integer solutions $(-1, 3)$ of $E_{(5,3;4)}$.

Let (x, y) be the integer solutions of $E_{(a,b;c)} : ax + by = c$. N denotes the number of times to need to amount c with the buckets a and b corresponding to these solutios (x, y). Then N is formulated as follows.

Theorem 3.2 (modified three jug problem) *Assume* $a > b$ *and* $\mathrm{GCD}(a, b) = \mathrm{GCD}(a, c) = \mathrm{GCD}(b, c) = 1$. *Then, using the length of the integer solution* $L(x, y) = |x| + |y|$, *the number of times* N *is expressed by;*

If $1 \le c < b$, *then* $N = 2L(x, y) - 2$.
If $b < c < a$ *and* $x > 0$, *then* $N = 2L(x, y) - 2$.
If $b < c < a$ *and* $x < 0$, *then* $N = 2L(x, y)$.

Assume $a > b > 0$ with $\mathrm{GCD}(a, b) = \mathrm{GCD}(a, c) = \mathrm{GCD}(b, c) = 1$ as above. Then, very roughly speaking, to determine the minimal number of times N for the above modified three jug problem is nothing but to determine the minimal length L_E of the equation $E_{(a,b;c)}$. Assume the additional condition $a > c > 0$, then there exists only one couple (x, y) of minimal integer solutions for the cases $c \neq \dfrac{a+b}{2}$ from Theorem 2.5. Moreover the case $c = \dfrac{a+b}{2}$ has exactly 2 minimal integer solutions;

$$(x, y) = \left(\frac{-b+1}{2}, \frac{a+1}{2}\right), \text{ and } \left(\frac{b+1}{2}, \frac{-a+1}{2}\right).$$

Theorem 3.3 *Let* a *be coprime positive integers with* $a > b$. *For any* c $(1 \le c < a)$, *the equation* $E_{(a,b;c)} : ax + by = c$ *has exactly one couple of minimal integer solutions except for the case* $c = \dfrac{a+b}{2}$. $E = E_{(a,b;(a+b)/2)} : ax + by = \dfrac{a+b}{2}$ *has exactly two minimal integer solutions*

$$(x, y) = \left(\frac{-b+1}{2}, \frac{a+1}{2}\right), \ \textit{and} \ \left(\frac{b+1}{2}, \frac{-a+1}{2}\right),$$

where the both minimal length is $L_E = \dfrac{a+b}{2}$.

Remark 3.4 *The exceptional case* $c = \dfrac{a+b}{2}$ *occurs only when* $a \equiv b \equiv 1 \pmod 2$.

Corollary 3.5 *Consider three jug problem for the case* a, b, *where* a *and* b *are coprime odd positive integers with* $a > b$. *Then the number of times* N *corresponding to the solutions* $\left(\dfrac{b+1}{2}, \dfrac{-a+1}{2}\right)$ *is* $a + b - 1$.

The number of times N *corresponding to the solutions* $\left(\dfrac{-b+1}{2}, \dfrac{a+1}{2}\right)$ *is* $a + b$.

Remark 3.6 *Three jug problem is sometimes called the decanter problem, where the liquid is wine. In Japan, Mitsuyoshi Yoshida published a book "Jinkouki" in 1627. In this book he treated three jug problem of the case* $[a, b] = [7, 3]$*, where the liquid is oil.*

3.3 Examples of minimal integer solutions

In this section, we shall treat the special class of equations $E_{(a,b;c)}$. Let F_n and L_n be n-th Fibonacci and Lucas numbers, respectively. Fibonacci numbers F_n and Lucas numbers L_n are defined by putting,

$$F_{n+1} = F_n + F_{n-1}, \text{ and } L_{n+1} = L_n + L_{n-1},$$

with inital terms $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$. For the sake of readers, we shall list Fibonacci numbers and Lucas numbers for small indices n.

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
F_n	1	0	1	1	2	3	5	8	13	21	34	55	89	144
L_n	-1	2	1	3	4	7	11	18	29	47	76	123	199	322

Here we will give the minimal solutions of the following equations for small c.

$$F_{n+1}x + F_n y = c, \text{ and } L_{n+1}x + L_n y = c, \text{where } 1 \leq c \leq 5.$$

The following well known formula is called Cassini's identity, which played the key role in our old paper [3].

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \text{i.e.,} \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

Thus one obtains

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = \begin{vmatrix} F_{n+1} & F_n - F_{n+1} \\ F_n & F_{n-1} - F_n \end{vmatrix} = \begin{vmatrix} F_{n+1} & -F_{n-1} \\ F_n & -F_{n-2} \end{vmatrix} = - \begin{vmatrix} F_{n+1} & F_{n-1} \\ F_n & F_{n-2} \end{vmatrix}.$$

Hence we have shown the equation $E_{(F_{n+1},F_n,1)} : F_{n+1}x + F_n y = 1$ have integer solutions $(x, y) = ((-1)^{n-1}F_{n-2}, (-1)^n F_{n-1})$. Actually these solutions are the minimal integer solutions and the minimal length is $L_E = F_n (= F_{n-1} + F_{n-2})$. Hence we have shown the following result.

(3.1) The equation $E_{(F_{n+1},F_n;1)}$
Minimal integer solutions are $(x, y) = ((-1)^{n-1}F_{n-2}, (-1)^n F_{n-1})$. The minimal length is $L_E = F_n$.
Similarly, one can easily verify the following examples.
(3.2) The equation $E_{(F_{n+1},F_n;2)}$
Minimal integer solutions are $x = (-1)^n F_{n-3}, y = (-1)^{n-1}F_{n-2}$. The minimal

length is $L_E = F_{n-1}$.
(3.3) The equation $E_{(F_{n+1},F_n;3)}$
Minimal integer solutions are $x = (-1)^{n-1}F_{n-4}, y = (-1)^n F_{n-3}$. The minimal length is $L_E = F_{n-2}$.
(3.4) The equation $E_{(F_{n+1},F_n;4)}$
Minimal integer solutions are $x = (-1)^n 2F_{n-3}, y = (-1)^{n-1}2F_{n-2}$, The minimal length is $L_E = 2F_{n-1}$.
(3.5) The equation $E_{(F_{n+1},F_n,;5)}$
Minimal integer solutions are $x = (-1)^n F_{n-5}, y = (-1)^{n-1}F_{n-4}$. The minimal length is $L_E = F_{n-3}$.
(3.6) The equation $E_{(L_{n+1},L_n;1)}$
Minimal integer solutions are $x = (-1)^n F_{n-1}, y = (-1)^{n+1}F_n$. The minimal length is $L_E = F_{n+1}$.
(3.7) The equation $E_{(L_{n+1},L_n;2)}$
Minimal integer solutions are $x = (-1)^{n+1}F_n, y = (-1)^n F_{n+1}$. The minimal length is $L_E = F_{n+2}$.
(3.8) The equation $E_{(L_{n+1},L_n;3)}$
Minimal integer solutions are $x = (-1)^{n-1}F_{n-2}, y = (-1)^n F_{n-1}$. The minimal length is $L_E = F_n$.
(3.9) The equation $E_{(L_{n+1},L_n;4)}$
Minimal integer solutionsa re $x = (-1)^n F_{n-3}, y = (-1)^{n-1}F_{n-2}$. The minimal length is $L_E = F_{n-1}$.
(3.10) The equation $E_{(L_{n+1},L_n;5)}$
Minimal integer solutions are $x = (-1)^n L_{n-2}, y = (-1)^{n-1}L_{n-1}$. The minimal length is $L_E = L_n$.

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