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On effective divisors on certain double covers and their linearly equivalent classes

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Abstract

Let $B \subset \mathbb{P}^2$ be a plane curve with even degree on the complex projective plane $\mathbb{P}^2$, and let $\phi : X \to \mathbb{P}^2$ be the double cover branched along B. In this paper, we compute ideals of certain divisors on X for certain smooth curves B of degree ≤ 4 without using rationality of X.

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1 Introduction

For two plane curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$, we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ have the *same embedded topology* if there is a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$. Let $\mathcal{C}_i = C_{i,1} + \cdots + C_{i,n_i}$ be the irreducible decomposition of a plane curve $\mathcal{C}_i \subset \mathbb{P}^2$ for each $i = 1, 2$. In the case where $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same embedded topology, it is known that the following conditions hold:

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(i) $n_1 = n_2 =: n$,

(ii) after relabeling $C_{2,1}, \ldots, C_{2,n}$ if necessary, the followings are satisfied:

(a) $h(C_{1,i}) = C_{2,i}$ for each $i = 1, \ldots, n$,

(b) $\deg C_{1,i} = \deg C_{2,i}$ for each $i = 1, \ldots, n$,

(c) the numbers and the topological types of singularities of $C_{1,i}$ are same with $C_{2,i}$ for each $i = 1, \ldots, n$.

(d) intersections of $C_{1,1}, \ldots, C_{1,n}$ are topologically same with those of $C_{2,1}, \ldots, C_{2,n}$.

One of problems on plane curves is to distinguish the embedded topology of two plane curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ satisfying the above conditions. The following theorem is used for this problem effectively.

Theorem 1.1 (cf. [4, Corollary 1.4]). *For each $i = 1, 2$, let $\mathcal{C}_i \subset \mathbb{P}^2$ be a plane curve consists of two irreducible components $B_i, C_i \subset \mathbb{P}^2$ with $\deg B_i = 2\ell$ for $\ell \in \mathbb{Z}_{>0}$. Let $\phi_i : X_i \to \mathbb{P}^2$ be the double cover branched along B_i for each $i = 1, 2$. If there is a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ with $h(B_1) = B_2$ and $h(C_1) = C_2$, then $s_{\phi_1}(C_1) = s_{\phi_2}(C_2)$, where $s_{\phi_i}(C_i)$ is the number of irreducible components of $\phi_i^* C_i$.*

With the same notation of Theorem 1.1, if $\deg C_1 = \deg C_2 \neq 2\ell$, and there is a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ with $h(\mathcal{C}_1) = \mathcal{C}_2$, then $h(B_1) = B_2$ and $h(C_1) = C_2$, and hence $s_{\phi_1}(C_1) = s_{\phi_2}(C_2)$.

Let $\phi : X \to \mathbb{P}^2$ be a double cover branched along $B \subset \mathbb{P}^2$. The number $s_\phi(C)$ is called the *splitting number* of C with respect to ϕ. In the case of $\deg B \neq \deg C$, Theorem 1.1 implies that the irreducibility of $\phi^* C$ is an invariant of embedded topology of $\mathcal{C} = B + C$. A criterion [4, Theorem 2.7] for irreducibility of $\phi^* C$ is given if C is smooth (cf. [2]). On the other hand, if C_i is singular, then such criterion is not known except for few cases (cf. [1]). In this paper, we consider an approach for the irreducibility of $\phi^* C$ by computing curves on the double cover X. Namely, we consider the following problem.

Problem 1.2. Let $B \subset \mathbb{P}^2$ be a plane curve of degree 2ℓ, and let $\phi : X \to \mathbb{P}^2$ be the double cover branched along B.

(i) Compute generators and relations of the divisor class group $\mathrm{Cl}(X)$.

(ii) For each curve $\overline{C} \subset X$, compute curves on X linearly equivalent to $\overline{C}$.

(iii) For each class $[\overline{C}] \in \mathrm{Cl}(X)$, give geometric characters (e.g. the arrangement of singularities) of the image $\phi(\overline{C})$.

If B is smooth with $\deg B = 2, 4$, then X is a rational surface, and $\mathrm{Cl}(X) = \mathrm{Pic}(X)$ is well known. On the other hand, it seems difficult to compute $\mathrm{Cl}(X)$ from data of B if $\deg B \geq 6$ in general. The aim of this paper is to compute curves on X linearly equivalent to certain curves $\overline{C} \subset X$ for $\deg B = 2, 4$ without using rationality of X.

Let $B \subset \mathbb{P}^2$ be a plane curve of degree 2ℓ, and let $F \in \mathbb{C}[x, y, z]$ be a defining polynomial of B. Let $\phi : X \to \mathbb{P}^2$ be the double cover branched along B. Then X can be regarded as the sub-variety in $\mathbb{P}(1, 1, 1, \ell)$ defined by $w^2 - F = 0$, where $\mathbb{P}(1, 1, 1, \ell)$ is the weighted projective space with weight $(1, 1, 1, \ell)$, and $[x : y : z : w]$ is a system of coordinates with $\deg x = \deg y = \deg z = 1$ and $\deg w = \ell$. Let R_X be the homogeneous coordinate ring $\mathbb{C}[x, y, z, w]/\langle w^2 - F\rangle$ of X:

$$X = \mathbb{V}(w^2 - F) \subset \mathbb{P}(1, 1, 1, \ell), \qquad R_X := \mathbb{C}[x, y, z, w]/\langle w^2 - F\rangle.$$

By abuse of notation, let f denote the class $[f]$ in R_X containing $f \in \mathbb{C}[x, y, z, w]$. For $d \in \mathbb{Z}_{\geq 0}$, let

$$(R_X)_d \subset R_X$$

denote the vector space over $\mathbb{C}$ generated by homogeneous elements of degree d. A prime (Weil) divisor $E \subset X$ defines a valuation $v_E : Q(R_X) \to \mathbb{Z} \cup \{\infty\}$ at E with $v_E(0) := \infty$ since R_X is normal (cf. [3, §9]), where $Q(R_X)$ is the quotient field of R_X. For an effective divisor $D = \sum_E n_E E$ on X, let $\mathbb{I}_X(D)$ be the ideal of R_X generated by homogeneous elements f such that $v_E(f) \geq n_E$ for any prime divisors E:

$$\mathbb{I}_X(D) := \left\langle f : \text{homog.} \;\middle|\; v_E(f) \geq n_E \text{ for } {}^\forall E \subset X\text{: prime} \right\rangle \subset R_X.$$

The main theorem of this paper is as follows.

The case of $\deg B = 2$**.** Put $F := z^2 + xy \in \mathbb{C}[x, y, z]$, and let $B \subset \mathbb{P}^2$ be the plane curve defined by $F = 0$. Let $\phi : X \to \mathbb{P}^2$ be the double cover branched along B. Then X is the sub-variety of $\mathbb{P}^3$ defined by $w^2 - F = 0$. Let $E^\pm \subset X$ be the curves defined by $w \pm z = x = 0$, respectively. Note that $E^\pm$ are prime divisors on X.

Proposition 1.3. *Let $m, n \in \mathbb{Z}_{\geq 0}$. The following equation holds:*

$$\mathbb{I}_X(nE^+ + mE^-) = \begin{cases} \left\langle x^{n-i}(w+z)^i \;\middle|\; i = 0, \dots, n-m \right\rangle & \text{if } n \geq m, \\ \left\langle x^{m-i}(w-z)^i \;\middle|\; i = 0, \dots, m-n \right\rangle & \text{if } n \leq m. \end{cases}$$

Theorem 1.4. *Let $m, n \in \mathbb{Z}_{\geq 0}$, and put $M_{\min} := \min(m, n)$. If an effective divisor D on X is linearly equivalent to $nE^+ + mE^-$, then there exist $h_0, \dots, h_{|n-m|} \in (R_X)_{M_{\min}}$ such that*

$$\mathbb{I}_X(D) = \left\langle \sum_{j=0}^{|n-m|} h_j x^{n-m-i-j}(w+z)^i(w-z)^j \;\middle|\; i = 0, \dots, |n-m| \right\rangle$$

An example of $\deg B = 4$. Let $F := x^4 + y^4 - z^4 \in \mathbb{C}[x, y, z]$, and let $B \subset \mathbb{P}^2$ be the quartic curve defined by $F = 0$. Let $\phi : X \to \mathbb{P}^2$ be the double cover branched along B. Let E_1 and E_2 be two prime divisors on X defined by the following equations:

$$E_1 : y - z = w + x^2 = 0, \qquad E_2 : x - z = w + y^2 = 0.$$

Put $\mathbb{I}^{(4)}_{n,m} := nE_1 + mE_2$. We obtain the following results.

Proposition 1.5. *Let $m, n \in \mathbb{Z}_{\geq 0}$, and put $t_1 := y - z$, $t_2 := x - z$, $t_3 := w + x^2 + y^2 - z^2$ in R_X. Then the following equation holds:*

$$\mathbb{I}^{(4)}_{n,m} = \begin{cases} \left\langle t_1^{n-i}t_2^{m-i}t_3^i,\ t_1^{n-j}t_3^j \;\middle|\; i = 0, \dots, m,\ j = m+1, \dots, n \right\rangle & \text{if } n \geq m, \\ \left\langle t_1^{n-i}t_2^{m-i}t_3^i,\ t_2^{m-j}t_3^j \;\middle|\; i = 0, \dots, n,\ j = n+1, \dots, m \right\rangle & \text{if } n \leq m. \end{cases}$$

Theorem 1.6. *Let $m, n \in \mathbb{Z}_{\geq 0}$ and $t_1, t_2, t_3 \in R_X$ be as Proposition 1.5. Let $M_{\min} := \min(m, n)$ and $M_{\max} := \max(m, n)$. Put*

$$A_{i,j} := t_1^{n-i}t_2^{m-i}t_3^{i-j}(x+z)^j(y+z)^j \quad \text{for } 0 \leq i \leq M_{\min} \text{ and } 0 \leq j \leq i,$$

$$B_{i,j} := t_1^{n-j}t_2^{m-j}(2w - t_3)^{j-i}(x+z)^i(y+z)^i \quad \text{for } \begin{cases} 0 \leq i \leq M_{\min} \\ i \leq j \leq M_{\min}, \end{cases}$$

$$A'_{i,j} := t_1^{n-i}t_3^{i-j}(x+z)^j(y+z)^j \quad \text{for } 0 \leq i \leq n \text{ and } m < j \leq n \text{ if } n > m,$$

$$A''_{i,j} := t_2^{m-i}t_3^{i-j}(x+z)^j(y+z)^j \quad \text{for } 0 \leq i \leq n \text{ and } n < j \leq m \text{ if } n < m.$$

Put $A'_{i,j} = A''_{i,j} = 0$ *if* $m = n$. *Then, for any divisor* D *on* X *linearly equivalent to* $nE_1 + mE_2$, *there exist* $c_j \in \mathbb{C}$ *for* $j = 0, \ldots, M_{\max}$ *such that* $\mathbb{I}_X(D)$ *is the following ideal of* R_X*:*

$$\begin{cases} \left\langle \sum_{j=0}^{i} c_j A_{i,j} + \sum_{j=i+1}^{m} (-2)^{i-j} c_j B_{i,j},\ \sum_{j=m+1}^{n} c_j A'_{i,j} \ \middle|\ i = 0, \ldots, m \right\rangle & \text{if } n \geq m, \\ \left\langle \sum_{j=0}^{i} c_j A_{i,j} + \sum_{j=i+1}^{n} (-2)^{i-j} c_j B_{i,j},\ \sum_{j=n+1}^{m} c_j A''_{i,j} \ \middle|\ i = 0, \ldots, n \right\rangle & \text{if } n \leq m. \end{cases}$$

2 Proofs

In this section, we give proofs of the main results. Let $\phi : X \to \mathbb{P}^2$ be a double cover branched along $B \subset \mathbb{P}^2$, and let $\iota : X \to X$ be the covering transformation of ϕ. Let $E^+ \subset X$ be a prime divisor with $E^+ \not\subset \phi^{-1}(B)$, and put

$$E^- := \iota^* E^+ \subset X, \qquad E := \phi(E^+) \subset \mathbb{P}^2.$$

Let R_{X,E^+} be the local ring of R_X at E^+, which is a DVR, and let $\mathfrak{m}_{X,E^+} \subset R_{X,E^+}$ be the maximal ideal.

Lemma 2.1. *If* $u_E \in H^0(\mathbb{P}^2, \mathcal{O}(E))$ *is a defining polynomial of* E, *then* $u_E \in R_{X,E^+}$ *is a uniformizing parameter of* R_{X,E^+}.

Proof. Let $f \in \mathbb{I}_X(E^+)$ be any homogeneous element; if $v_{E^-}(f) \geq 1$, then $f \in \langle u_E \rangle \subset R_X$ since $E^+ \not\subset \phi^{-1}(B)$; if $v_{E^-}(f) = 0$, then $\iota^* f \notin \mathbb{I}_X(E^+)$ and $f \cdot \iota^* f \in \langle u_E \rangle$, hence there is $h \in R_{X,E}$ such that $f = hu_E/\iota^* f$. Thus $\mathfrak{m}_{X,E^+}$ is generated by u_E in R_{X,E^+}. □

Lemma 2.2. *For two effective divisors* $D = \sum n_E E$, $D' = \sum n'_E E$, *if* D *and* D' *are linearly equivalent,* $D \sim D'$, *then there is a rational function* $q \in \mathbb{C}(X)^\times$ *such that* $\mathbb{I}_X(D) = q \cdot \mathbb{I}_X(D')$.

Proof. Since $D \sim D'$, there is a rational function $q \in \mathbb{C}(X)^\times$ such that $D - D' = (q)$, where (q) is the principal divisor on X defined by q. Then we have $f'q \in \mathbb{I}_X(D)$ for any $f' \in \mathbb{I}_X(D')$ and any prime divisor E on X since $v_E(f'q) \geq n_E$. Similarly, we have $fq^{-1} \in \mathbb{I}_X(D')$ for any $f \in \mathbb{I}_X(D)$. Therefore $\mathbb{I}_X(D) = q \cdot \mathbb{I}_X(D')$. □

2.1 Proof of Theorem 1.4

Let $B \subset \mathbb{P}^2$ be the smooth conic defined by $F := z^2 + xy = 0$, and let $\phi : X \to \mathbb{P}^2$ be the double cover branched along B. We can regard X and ϕ as the sub-variety of $\mathbb{P}^3$ defined by $w^2 - F = 0$ and the map $X \to \mathbb{P}^2$ given by $\phi(x : y : z : w) := [x : y : z]$, respectively. Let $E^{\pm}$ be prime divisors on X defined by $x = w \pm z = 0$, respectively:

$$E^+ \subset X : x = w + z = 0, \qquad E^- \subset X : x = w - z = 0.$$

Lemma 2.3. *For each $m \in \mathbb{Z}_{\geq 0}$, $\mathbb{I}_X(mE^+ + mE^-) \subset R_X$ is the ideal generated by x^m:*

$$\mathbb{I}_X(mE^+ + mE^-) = \langle x^m \rangle \subset R_X.$$

Proof. Put $\mathbb{I}_m := \mathbb{I}_X(mE^+ + mE^-)$. It is clear that $\mathbb{I}_0 = R_X = \langle x^0 \rangle$. Let $L_x \subset \mathbb{P}^2$ be the line defined by $x = 0$. Since $\phi^* mL_x = mE^+ + mE^-$, we have $\mathbb{I}_m = \langle x^m \rangle$. □

Lemma 2.4. *For each $m \in \mathbb{Z}_{\geq 0}$, $\mathbb{I}_X(mE^+)$ is the ideal of R_X generated by $x^{m-i}(w+z)^i$ for $i = 0, \dots, m$:*

$$\mathbb{I}_X(mE^+) = \langle x^{m-i}(w+z)^i \mid i = 0, \dots, m \rangle.$$

Proof. Put $\mathbb{I}_m := \mathbb{I}_X(mE^+)$, and $I_m := \langle x^{m-i}(w+z)^i \mid i = 0, \dots, m \rangle$. We prove the following claim.

Claim 2.5. *Let k be an integer with $0 \leq k \leq m-1$. If $h_{k,i} \in R_X$ for $i = k, \dots, m-1$ satisfies*

$$f_k := \sum_{i=k}^{m-1} h_{k,i} x^{m-i-1}(w+z)^i \equiv 0 \pmod{\mathbb{I}_m},$$

then there are $h_{k+1,j} \in R_X$ for $j = k+1, \dots, m-1$ such that

$$f_{k+1} := \sum_{i=k+1}^{m-1} h_{k+1,i} x^{m-i-1}(w+z)^i \equiv f_k \pmod{I_m}.$$

Proof of Claim 2.5. Let R_{X,E^+} be the local ring at $\mathbb{I}_1 = \mathbb{I}_x(E^+)$. Note that the maximal ideal $\mathfrak{m}_{X,E^+} \subset R_{X,E^+}$ is generated by x. Since $w^2 - z^2 = xy$, we obtain

$$f_k = \frac{x^{m-1}}{(w-z)^{m-1}} \sum_{i=k}^{m-1} h_{k,i} y^i (w-z)^{m-i-1}$$

as elements of R_{X,E^+}. Since $\mathfrak{m}_{X,E^+} \cap R_X = \mathbb{I}_1$ and $f_k \in \mathbb{I}_m$, we obtain

$$\sum_{i=k}^{m-1} h_{k,i} y^i (w-z)^{m-i-1} \in \mathbb{I}_1.$$

Since $x, w+z \in \mathbb{I}_1$, there are $h'_{k,i} \in \mathbb{C}[y,z]$ such that $h'_{k,i} \equiv h_{k,i} \pmod{\mathbb{I}_1}$. Moreover, we have

$$0 \equiv \sum_{i=k}^{m-1} h_{k,i} y^i (w-z)^{m-i-1} \equiv \sum_{i=k}^{m-1} h'_{k,i} y^i (-2z)^{m-i-1} \pmod{\mathbb{I}_1}.$$

Since $R_X/\mathbb{I}_1 \cong \mathbb{C}[y,z]$, we have

$$h'_{k,k}(-2z)^{m-k-1} + h'_{k,k+1} y^{k+1} (-2z)^{m-k-2} + \cdots + h'_{k,m-1} y^{m-1} = 0$$

as polynomials in $\mathbb{C}[y,z]$. Hence there is $g_{k,k} \in \mathbb{C}[y,z]$ such that $h'_{k,k} = y g_{k,k}$. Since $x^{m-i}(w+z)^i, x^{m-i-1}(w+z)^{i+1} \in I_m$, we have in R_X/I_m

$$\begin{aligned} f_k &= \sum_{i=k}^{m-1} h_{k,i} x^{m-i-1}(w+z)^i \equiv \sum_{i=k}^{m-1} h'_{k,i} x^{m-i-1}(w+z)^i \\ &\equiv g_{k,k}(xy) x^{m-k-2}(w+z)^k + h'_{k,k+1} x^{m-k-2}(w+z)^{k+1} \\ &\qquad + \cdots + h_{k,m-1}(w+z)^{m-1} \\ &\equiv g_{k,k}(w^2-z^2) x^{m-k-2}(w+z)^k + h'_{k,k+1} x^{m-k-2}(w+z)^{k+1} \\ &\qquad + \cdots + h_{k,m-1}(w+z)^{m-1}. \end{aligned}$$

Since $w^2 - z^2 = -2z(w+z) + (w+z)^2$, by putting

$$\begin{aligned} h_{k+1,k+1} &:= -2z g_{k,k} + h'_{k,k+1}, \\ h_{k+1,k+2} &:= x g_{k,k} + h'_{k,k+2}, \\ h_{k+1,j} &:= h_{k,j} \qquad (j = k+3, \ldots, m-1), \end{aligned}$$

we obtain $f_k \equiv f_{k+1} \pmod{I_m}$. □

Let us return to the proof of Lemma 2.4. If $m = 0, 1$, the equation $\mathbb{I}_m = I_m$ is clear. Suppose that $m > 1$ and $\mathbb{I}_{m-1} = I_{m-1}$. By the definition of $\mathbb{I}_m$, we have $\mathbb{I}_m \supset I_m$. Let $f \in \mathbb{I}_m$ be any homogeneous element of degree d.

Since $\mathbb{I}_m \subset \mathbb{I}_{m-1} = I_{m-1}$, there are homogeneous elements $h_i \in (R_X)_{d-m+1}$ for $i = 0, \ldots, m-1$ such that

$$f = \sum_{i=0}^{m-1} h_i x^{m-i-1}(w+z)^i.$$

Put $h_{0,i} := h_i$ for $i = 0, \ldots, m-1$, and $f_0 := f$. With the notation of Claim 2.5, we obtain

$$f = f_0 \equiv f_1 \equiv \cdots \equiv f_{m-1} = h_{m-1,m-1}(w+z)^{m-1} \pmod{I_m}.$$

Since $f \in \mathbb{I}_m$ and $\mathbb{I}_m \supset I_m$,

$$m \le v_{E^+}(h_{m-1,m-1}(w+z)^{m-1}).$$

Thus we have $v_{E^+}(h_{m-1,m-1}) \ge 1$, and $h_{m-1,m-1} \in \mathbb{I}_1 = xR_X + (w+z)R_X$. Therefore $f \equiv h_{m-1,m-1}(w+z)^{m-1} \equiv 0 \pmod{I_m}$. □

By the same argument, we can prove the following lemma.

Lemma 2.6. *For each $m \in \mathbb{Z}_{\ge 0}$, the following equation holds:*

$$\mathbb{I}_X(mE^-) = \langle x^{m-i}(w-z)^i \mid i = 0, \ldots, m \rangle \subset R_X.$$

We are ready to prove Proposition 1.3.

Proof of Proposition 1.3. Put $\mathbb{I}_{n,m} := \mathbb{I}_X(nE^+ + mE^-)$. We first suppose that $n \ge m$. Put

$$I^+_{n,m} := \langle x^{n-i}(w+z)^i \mid i = 0, \ldots, n-m \rangle \subset R_X.$$

Let $f \in I^+_{n,m}$ be a homogeneous element. Since $v_{E^\pm}(x^m) = m$ and $v_{E^+}(w + z) = 1$, we have $v_{E^+}(f) \ge n$, $v_{E^-}(f) \ge m$, and hence $f \in \mathbb{I}^+_{n,m}$.

Conversely, let $f \in \mathbb{I}^+_{n,m}$ be a homogeneous element. Since $v_{E^+}(f) \ge n \ge m$ and $v_{E^-}(f) \ge m$, there is a homogeneous element $g \in R_X$ such that $f = x^m g$. Then

$$n \le v_{E^+}(f) = v_{E^+}(x^m) + v_{E^+}(g) = m + v_{E^+}(g).$$

Thus we have $v_{E^+}(g) \ge n - m$. Since

$$f \in \langle x^{n-m-i}(w+z)^i \mid i = 0, \ldots, n-m \rangle$$

by Lemma 2.4, there are homogeneous elements $h_i \in R_X$ such that

$$g = \sum_{i=0}^{n-m} h_i x^{n-m-i}(w+z)^i.$$

Therefore $f \in I^+_{n,m}$, and $\mathbb{I}^+_{n,m} = I^+_{n,m}$.

In the case of $n \leq m$, we can prove the assertion by the same argument using Lemma 2.3 and 2.6. □

Let $S_X \subset R_X$ be the set of all homogeneous elements, which is a multiplicatively closed set. Note that the rational function field $\mathbb{C}(X)$ of X can be regarded as the sub-field $(S_X^{-1}R_X)_0$ of the localized ring $S_X^{-1}R_X$ consisting of homogeneous elements of degree 0 and the zero element.

Proposition 2.7. *Let $n, m \in \mathbb{Z}_{\geq 0}$, and put $M_{\min} := \min(n, m)$. For $q \in \mathbb{C}(X)^\times$, $q \cdot \mathbb{I}_X(nE^+ + mE^-) \subset R_X$ if and only if there are $h_0, \ldots, h_{|n-m|} \in (R_X)_{M_{\min}}$ such that*

$$q = \begin{cases} \displaystyle\sum_{i=0}^{n-m} \frac{h_i}{x^m}\left(\frac{w-z}{x}\right)^i & \text{if } n \geq m, \\ \displaystyle\sum_{i=0}^{m-n} \frac{h_i}{x^n}\left(\frac{w+z}{x}\right)^i & \text{if } n \leq m. \end{cases}$$

To prove Proposition 2.7, we prove the following lemma.

Lemma 2.8. *Let $k, n \in \mathbb{Z}_{\geq 0}$ with $0 \leq k \leq n$, and let $q \in \mathbb{C}(X)^\times$. If $qx^{n-j}(w+z)^j \in R_X$ for each $j = 0, \ldots, k$, then there are homogeneous polynomials $a_0 \in \mathbb{C}[x, y, z]_{n-k}$, $b_0 \in \mathbb{C}[x, y, z]_{n-k-1}$ and $a'_i \in \mathbb{C}[y, z]_{n-k}$ for $i = 1, \ldots, k$ such that*

$$q = \frac{a_0 + b_0 w}{x^{n-k}} + \sum_{i=1}^{k} \frac{a'_i}{x^{n-k}}\left(\frac{w-z}{x}\right)^i. \tag{2.1}$$

Proof. We prove the assertion by the induction on k. In the case of $k = 0$, $qx^n \in R_X$ if and only if there is a homogeneous polynomials $a_0, b_0 \in R_X$ of degree n and $n-1$, respectively, such that $q = (a_0 + b_0 w)/x^n$.

Suppose that $k \geq 1$. By the assumption of the induction, there are $a_0 \in \mathbb{C}[x,y,z]_{n-k+1}$, $b_0 \in \mathbb{C}[x,y,z]_{n-k}$ and $a_i' \in \mathbb{C}[y,z]_{n-k+1}$ for $i = 1,\ldots,k-1$ such that

$$q = \frac{a_0 + b_0 w}{x^{n-k+1}} + \sum_{i=1}^{k-1} \frac{a_i'}{x^{n-k+1}} \left(\frac{w-z}{x}\right)^i.$$

Let $a_0', b_0' \in \mathbb{C}[y,z]$ and $a_0', b_0' \in \mathbb{C}[x,y,z]$ be the homogeneous polynomials such that

$$a_0 = a_0'' x + a_0', \qquad b_0 = b_0'' x + b_0'.$$

We consider the R_X-module $x^{-n}R_X$ and its quotient module $(x^{-n}R_X)/R_X$. Since $(w+z)^2 = 2z(w+z) + xy$, we have

$$(w+z)^i \equiv (2z)^{i-1}(w+z) \pmod{xR_X}.$$

for each $i \geq 1$. Since $a_0 + b_0 w = (a_0 + b_0 z) + b_0(w-z)$ and $w^2 - z^2 = xy$,

$$(a_0 + b_0 w)(w+z) \equiv (a_i + b_i z)(w+z) \pmod{xR_X}.$$

By $qx^{n-k}(w+z)^k \in R_X$, we have in $x^{-n}R_X/R_X$

$$0 \equiv qx^{n-k}(w+z)^k \equiv \frac{1}{x}\left((2z)^{k-1}(a_0' + b_0' z) + \sum_{i=1}^{k-1} a_i' y^i (2z)^{k-i-1}\right)(w+z)$$

Let $q_0' \in \mathbb{C}[y,z]$ be the element

$$q_0' := (2z)^{k-1}(a_0' + b_0' z) + \sum_{i=1}^{k-1} a_i' y^i (2z)^{k-i-1}.$$

The above computation implies that $q_0' \in xR_X$. Since $q' \in \mathbb{C}[y,z]$, we obtain $q_0' = 0$. Thus there is $b_1' \in \mathbb{C}[y,z]$ of degree $n-k$ such that $a_0' = yb_1' - zb_0'$. Then we obtain

$$\frac{a_0' + b_0' w}{x^{n-k+1}} = \frac{b_0'}{x^{n-k}}\left(\frac{w-z}{x}\right) + \frac{yb_1'}{x^{n-k+1}},$$

$$q_1' := (2z)^{k-2}(a_1' + 2zb_1') + \sum_{i=2}^{k-1} a_i' y^{i-1}(2z)^{k-i-1} = 0.$$

We assume that there is $b'_j \in \mathbb{C}[y,z]$ of $n-k$ for $j = 1, \ldots, i$ $(i < k-1)$ such that

$$a'_j = yb'_{j+1} - 2zb'_j \qquad (j = 1, \ldots, i-1),$$
$$q'_j := (2z)^{k-j-1}(a'_j + 2zb'_j) + \sum_{i=j+1}^{k-1} a'_i y^{i-j}(2z)^{k-i-1} = 0 \qquad (j = 1, \ldots, i).$$

By $q'_i = 0$, there is $b'_{i+1} \in \mathbb{C}[y,z]$ of degree $n-k$ such that

$$a'_i = yb'_{i+1} - 2zb'_i,$$
$$q'_{i+1} := (2z)^{k-i-2}(a'_{i+1} + 2zb'_i) + \sum_{s=i+2}^{k-1} a'_s y^{s-i-1}(2z)^{k-s-1} = 0.$$

Since $(w-z)^2 = xy - 2(w-z)$, we obtain

$$\frac{yb'_i}{x^{n-k+1}}\left(\frac{w-z}{x}\right)^{i-1} + \frac{a'_i}{x^{n-k+1}}\left(\frac{w-z}{x}\right)^{i} = \frac{b'_i}{x^{n-k}}\left(\frac{w-z}{x}\right)^{i} + \frac{yb'_{i+1}}{x^{n-k+1}}\left(\frac{w-z}{x}\right)^{i}$$

Since $q'_{k-1} = a'_{k-1} + 2zb'_{k-1} = 0$,

$$\frac{yb'_{k-1}}{x^{n-k+1}}\left(\frac{w-z}{x}\right)^{k-2} + \frac{a'_{k-1}}{x^{n-k+1}}\left(\frac{w-z}{x}\right)^{k-1} = \frac{b'_{k-1}}{x^{n-k}}\left(\frac{w-z}{x}\right)^{k}$$

The above argument proves the assertion. □

Proof of Proposition 2.7. Suppose that $n \geq m$. If $q = x^{-m}\sum_{i=0}^{n-m} h_i x^{-i}(w-z)^i$ for some homogeneous elements $h_i \in (R_X)_m$, then, for each $j = 0, \ldots, n-m$,

$$\begin{aligned} qx^{n-j}(w+z)^j &= \sum_{i=0}^{n-m} h_i x^{n-m-i-j}(w-z)^i(w+z)^j \\ &= \sum_{i=0}^{n-m} h_i x^{n-m-i-j}(xy)^{\min(i,j)}(w+\varepsilon_{i,j}z)^{|i-j|} \in R_X, \end{aligned}$$

where $\varepsilon_{i,j} = 1$ if $i \leq j$, and $\varepsilon_{i,j} = -1$ otherwise. Hence $q \cdot \mathbb{I}_X(nE^+ + mE^-) \subset R_X$ by Proposition 1.3.

Conversely, if $q\cdot\mathbb{I}_X(nE^+ + mE^-) \subset R_X$, then $qx^{n-j}(w+z)^j \in R_X$ for each $j = 0, \dots, n-m$ by Proposition 1.3; and there are $h_0, \dots, h_{n-m} \in (R_X)_m$ such that

$$q = \sum_{i=0}^{n-m} \frac{h_i}{x^m} \left(\frac{w-z}{x}\right)^i.$$

This prove the assertion in the case of $n \geq m$. We can prove this proposition in the case of $n < m$ by the same argument. We omit the details here. □

Next we prove Thorem 1.4.

Proof of Theorem 1.4. Assume that D be an effective divisor on X linearly equivalent to $nE^+ + mE^-$. Then there exists a rational function $q \in \mathbb{C}(X)$ such that

$$\mathbb{I}_X(D) = q \cdot \mathbb{I}_X(nE^+ + mE^-).$$

Suppose that $n \geq m$. Since

$$x^{n-i}(w+z)^i \sum_{j=0}^{n-m} \frac{h_j}{x^m} \left(\frac{w-z}{x}\right)^j = \sum_{j=0}^{n-m} h_j x^{n-m-i-j}(w+z)^i (w-z)^j$$

for homogeneous elements $h_j \in R_X$ of degree m, the assertion follows from Proposition 1.3 and 2.7. We can prove the assertion in the case of $n < m$ by the same argument. □

2.2 An example of $\deg B = 4$

Put $F := x^4 + y^4 - z^4$, and let $B \subset \mathbb{P}^2$ be the plane curve defined by $F = 0$. Let $\phi : X \to \mathbb{P}^2$ be the double cover branched along B, and we regard X as the sub-variety of $\mathbb{P}(1,1,1,2)$ defined by $w^2 - F = 0$. Let E_1, E_2 be the divisors on X defined by $y - z = w + x^2 = 0$ and $x - z = w + y^2 = 0$, respectively:

$$E_1 \subset X \ : \ y - z = w + x^2 = 0, \qquad E_2 \subset X \ : \ x - z = w + y^2 = 0.$$

Note that E_1 and E_2 are prime divisors on X. Put

$$t_1 := y - z, \qquad t_2 := x - z, \qquad t_3 := w + x^2 + y^2 - z^2$$

$$I_{n,m} := \begin{cases} \left\langle t_1^{n-i} t_2^{m-i} t_3^i,\ t_1^{n-j} t_3^j \ \middle|\ i = 0, \dots, m,\ j = m+1, \dots, n \right\rangle & \text{if } n \geq m, \\ \left\langle t_1^{n-i} t_2^{m-i} t_3^i,\ t_2^{m-j} t_3^j \ \middle|\ i = 0, \dots, n,\ j = n+1, \dots, m \right\rangle & \text{if } n \leq m. \end{cases}$$

Lemma 2.9. *Let $n \in \mathbb{Z}_{\geq 0}$. the following equations hold:*

$$\mathbb{I}_X(nE_1) = I_{n,0}, \qquad \mathbb{I}_X(nE_2) = I_{0,n}.$$

Proof. We prove $\mathbb{I}_X(nE_1) = I_{n,0}$ by the induction on n. We have $\mathbb{I}_X(0) = I_{0,0} = R_X$ and $\mathbb{I}_X(E_1) = I_{1,0}$. Suppose that $n > 1$, and $f \in \mathbb{I}_X(nE_1)$. By the assumption of the induction, there are $h_0, \ldots, h_{n-1} \in R_X$ such that

$$f = \sum_{i=0}^{n-1} h_i t_1^{n-i-1} t_3^i.$$

On the local ring R_{X,E_1} at E_1, we have

$$\begin{aligned}
f &= \sum_{i=0}^{n-1} h_i (y-z)^{n-i-1} \left(\frac{y^4 - z^4}{w - x^2} + y^2 - z^2 \right)^i \\
&= \sum_{i=0}^{n-1} h_i (y-z)^{n-i-1} \left(\frac{(y^2+z^2)(y^2-z^2) + (y+z)(w-x^2)}{w - x^2} \right)^i \\
&= \frac{t_1^{n-1}}{(w-x^2)^{n-1}} \sum_{i=0}^{n-1} h_i (w-x^2)^{n-i-1} (w - x^2 + y^2 + z^2)^i (y+z)^i.
\end{aligned}$$

Since $(y-z)R_{X,E_1}$ is the maximal ideal of $R_{X.E_1}$ by Lemma 2.1, we have $(y-z)R_{X,E_1} \cap R_X = \mathbb{I}_X(E_1)$. Since $v_{E_1}(f) \geq n$ by $f \in \mathbb{I}_X(nE_1)$,

$$f' := \sum_{i=0}^{n-1} h_i (w-x^2)^{n-i-1} (w - x^2 + y^2 + z^2)^i (y+z)^i \in \mathbb{I}_X(E_1).$$

Since $z \equiv y$, $w \equiv -x^2 \pmod{\mathbb{I}_X(E_1)}$, there are $h'_i \in \mathbb{C}[x,y]$ such that $h'_i \equiv h_i \pmod{\mathbb{I}_X(E_1)}$ for $i = 0, \ldots, n-1$. Hence we obtain, on $R_X/\mathbb{I}_X(E_1)$,

$$\begin{aligned}
0 \equiv f' &\equiv \sum_{i=0}^{n-1} h'_i (-2x^2)^{n-i-1} (-2x^2 + 2y^2)^i (2y)^i \\
&\equiv \sum_{i=0}^{n-1} h'_{k,i} (-2x^2)^{n-i-1} \Big(-4y(x^2-y^2) \Big)^i.
\end{aligned}$$

Since $R_X/\mathbb{I}_X(E_1) \cong \mathbb{C}[x,y]$,

$$\sum_{i=0}^{n-1} h'_i (-2x^2)^{n-i-1} \Big(-4y(x^2-y^2) \Big)^i = 0 \tag{2.2}$$

as a polynomial in $\mathbb{C}[x,y]$. Thus there is $g_{n-1} \in \mathbb{C}[x,y]$ such that

$$h'_{n-1} = -2x^2 g_{n-1} = -\Big(t_3 - (y+z)t_1\Big) g_{n-1} + (w - x^2) g_{n-1}.$$

By (2.2) again, there is $g_{k,n-2} \in \mathbb{C}[x,y]$ such that

$$h'_{k,n-2} = -2x^2 g_{k,n-2} + 4y(x^2 - y^2) g_{k,n-1}.$$

Since $\alpha\beta \in I_{n,0}$ for any $\alpha \in I_{n-1,0}$ and $\beta \in \mathbb{I}_X(E_1)$, and $t_3(w - x^2) = t_1(y+z)(w - x^2 + y^2 + z^2)$ on R_X, we obtain on $R_X/I_{n,0}$

$$\begin{aligned} f &\equiv g_{n-1}(-2x^2)t_3^{n-1} + \sum_{i=0}^{n-2} h'_i t_1^{n-i-1} t_3^i \\ &\equiv g_{n-1}\Big((w - x^2)t_3 + 4y(x^2 - y^2)t_1\Big) t_3^{n-2} \\ &\quad + g_{n-2}(-2x^2)t_1 t_3^{n-2} + \sum_{i=0}^{n-3} h'_i t_1^{n-i-1} t_3^i \\ &\equiv g_{n-1}(y - z)\Big((y+z)(w - x^2 + y^2 + z^2) - 4y(x^2 - y^2)\Big) t_3^{n-2} \\ &\quad + g_{n-2}(-2x^2)t_1 t_3^{n-2} + \sum_{i=0}^{n-3} h'_i t_1^{n-i-1} t_3^i \\ &\equiv g_{n-2}(-2x^2)t_1 t_3^{n-2} + \sum_{i=0}^{n-3} h'_i t_1^{n-i-1} t_3^i. \end{aligned}$$

By repeating this operation as the proof of Proposition 1.3, $\mathbb{I}_X(nE_1) \subset I_{n,0}$ can be proved. The inclusion $\mathbb{I}_X(nE_1) \supset \mathbb{I}_{n,0}$ is trivial. Hence we obtain $\mathbb{I}_X(nE_1) = I_{n,0}$. By the same argument, we can prove $\mathbb{I}_X(nE_2) = I_{0,n}$. $\square$

Proof of Proposition 1.5. We prove $\mathbb{I}_X(nE_1 + mE_2) = I_{n,m}$ in the case of $n \geq m$. Let f be a homogeneous element of $I_{n,m}$. Then there are $h_0, \ldots, h_n \in R_X$ such that

$$f = \sum_{i=0}^{m} h_i t_1^{n-i} t_2^{m-i} t_3^i + \sum_{i=m+1}^{n} h_i t_1^{n-i} t_3^i.$$

Note that we have

$$\begin{aligned} (w - x^2)t_3 &= (y+z)(w - x^2 + y^2 + z^2)t_1, \\ (w - y^2)t_3 &= (x+z)(w + x^2 - y^2 + z^2)t_2, \\ v_{E_1}(t_3) &= v_{E_2}(t_3) = 1. \end{aligned}$$

Thus $v_{E_1}(f) \geq n$, $v_{E_2}(f) \geq m$, and hence $f \in \mathbb{I}_X(nE_1 + mE_2)$.

We prove $\mathbb{I}_X(nE_1 + mE_2) = I_{n,m}$ by the induction on m. Recall that $t_1 := y - z$, $t_2 := x - z$, $t_3 := w + x^2 + y^2 - z^2$. By Lemma 2.9, the above equation holds when $m = 0$. Suppose that $m > 0$, and $f \in \mathbb{I}_X(nE_1 + mE_2)$. By the assumption of the induction, there are $h_0, \ldots, h_n \in R_X$ such that

$$f = \sum_{i=0}^{m-1} h_i t_1^{n-i} t_2^{m-i-1} t_3^i + \sum_{i=m}^{n} h_i t_1^{n-i} t_3^i.$$

On the local ring R_{X,E_2}, we have

$$f \equiv \frac{t_2^{m-1}}{(w-y^2)^{m-1}} \sum_{i=0}^{m-1} h_i t_1^{n-i} (x+z)^i (w-y^2)^{m-i-1} (w + x^2 - y^2 + z^2)^i$$

modulo $t_2^m R_{X,E_2}$. Since $f \in t_2^m R_{X,E_2}$, we obtain

$$\sum_{i=0}^{m-1} h_i t_1^{n-i} (x+z)^i (w-y^2)^{m-i-1} (w + x^2 - y^2 + z^2)^i \in t_2 R_{X,E_2} \cap R_X = \mathbb{I}_X(E_2).$$

Since $z \equiv x$, $w \equiv -y^2 \pmod{\mathbb{I}_X(E_2)}$, there are $h_i' \in \mathbb{C}[x,y]$ such that $h_i \equiv h_i'$ $(\mathrm{mod}\ \mathbb{I}_X(E_2))$ for $i = 0, \ldots, m-1$. Moreover,

$$\begin{aligned} 0 &\equiv \sum_{i=0}^{m-1} h_i t_1^{n-i} (x+z)^i (w-y^2)^{m-i-1} (w + x^2 - y^2 + z^2)^i \\ &\equiv \sum_{i=0}^{m-1} h_i' (y-x)^{n-i} (-2y^2)^{m-i-1} (2x)^i (2x^2 - 2y^2)^i \pmod{\mathbb{I}_X(E_2)}. \end{aligned}$$

Since $R_X/\mathbb{I}_X(E_2) \cong \mathbb{C}[x,y]$, we have

$$\sum_{i=0}^{m-1} h_i' (y-x)^{n-i} (-2y^2)^{m-i-1} (2x)^i (2x^2 - 2y^2)^i = 0. \tag{2.3}$$

Hence there is $g_{h_{m-1}} \in \mathbb{C}[x,y]$ such that

$$h_{m-1}' = -2y^2(y-x) g_{m-1}.$$

By (2.3), there is $g_{m-2} \in \mathbb{C}[x,y]$ such that

$$h_{m-2}' = -2y^2(y-x) g_{m-2} - 2x(2x^2 - 2y^2) g_{m-1}.$$

Since $\alpha\beta \in I_{n,m}$ for any $\alpha \in I_{n,m-1}$ and $\beta \in \mathbb{I}_X(E_2)$, and

$$\begin{aligned}-2y^2t_3 &= (w-y^2)(w+x^2+y^2-z^2)-(w+y^2)(w+x^2+y^2-z^2)\\ &= (x+z)(w+x^2-y^2+z^2)t_2-(w+y^2)t_3\\ &= (x+z)(t_3-2y^2+2z^2)t_2-(t_3-x^2+z^2)t_3\\ &= 2(x+z)t_2t_3-2(x+z)(y+z)t_1t_2-t_3^2,\end{aligned}$$

on R_X, we obtain on $R_X/I_{n,m}$

$$\begin{aligned}&h_{m-1}t_1^{n-m+1}t_3^{m-1}+h_{m-2}t_1^{n-m+2}t_2t_3^{m-2}\\ \equiv&-2y^2(y-x)t_1^{n-m+2}t_2t_3^{m-2}g_{m-2} \pmod{I_{n,m}}.\end{aligned}$$

By repeating this operation as the proof of Proposition 1.3, $\mathbb{I}_X(nE_1+mE_2) \subset I_{n,m}$ can be proved. The inclusion $\mathbb{I}_X(nE_1+mE_2) \supset \mathbb{I}_{n,m}$ is trivial. Hence we obtain $\mathbb{I}_X(nE_1+mE_2) = I_{n,m}$ if $n \geq m$. By the same argument, we can prove $\mathbb{I}_X(nE_1+mE_2) = I_{n,m}$ in the case of $n \leq m$. □

From now we prove Theorem 1.6 in the case of $n \geq m$. Note that we can prove it in the case of $n < m$ by the same argument. We first prove the following proposition.

Proposition 2.10. *Let $n, m \in \mathbb{Z}_{\geq 0}$, and put $M_{\min} := \min(n,m)$. For $q \in \mathbb{C}(X^\times)$, $q \cdot \mathbb{I}_X(nE_1+mE_2) \subset R_X$ if and only if there are $c_0, \ldots, c_{M_{\min}} \in \mathbb{C}$ such that*

$$q = \sum_{i=0}^{M_{\min}} c_i \frac{(x+z)^i(y+z)^i}{(w+x^2+y^2-z^2)^i}.$$

We prove this proposition in the case of $n \geq m$. By Proposition 1.5, we have

$$\mathbb{I}_X(nE_1+mE_2) = \langle t_1^{n-i}t_2^{m-i}t_3^i, t_1^{n-j}t_3^j \mid i=0,\ldots,m,\ j=m+1,\ldots,n\rangle,$$

where $t_1 := y-z$, $t_2 := x-z$, $t_3 := w+x^2+y^2-z^2$. In order to prove the above proposition, put $Q(R_X) := \mathbb{C}(x,y,z)[w]/\langle w^2-F\rangle$, which is the quotient field of R_X. We call $q \in Q(R_X)$ a *homogeneous element* if there are homogeneous elements $q', q'' \in R_X$ with $q'' \neq 0$ such that $q = q'/q''$, and put $\deg q := \deg q' - \deg q''$. Note that $\mathbb{C}(X)$ can be regarded as the $\mathbb{C}$-vector space $Q(R_X)_0 \subset Q(R_X)$ generated by homogeneous elements of degree 0.

Lemma 2.11. *Let $q \in Q(R_X)$ be a homogeneous element of degree d. If $q \cdot t_1^j t_2^j t_3^{m-j} \in R_X$ for any $j = 0, \dots, k$, then there are $c_0, \dots, c_k \in (R_X)_{d+2(m-k)}$ such that*

$$q = \frac{1}{t_3^{m-k}} \sum_{i=0}^{k} c_i \frac{(x+z)^i (y+z)^i}{t_3^i}.$$

Proof. We prove this lemma by the induction on k. If $k = 0$, then $c_0 := q \cdot t_3^m \in R_X$ is a homogeneous element of degree $d + m$ with $q = c_0 t_3^{-m}$. Suppose that $k > 0$, and $q \cdot t_1^j t_2^j t_3^{m-j} \in R_X$ for each $j = 0, \dots, k$. By the assumption of the induction, there are $c_0, \dots, c_{k-1} \in (R_X)_{d+(m-k+1)}$ such that

$$q = \frac{1}{t_3^{m-k+1}} \sum_{i=0}^{k-1} c_i \frac{(x+z)^i (y+z)^i}{t_3^i}.$$

Let $\alpha_i \in (R_X)_{d+2(m-k)}$ and $\beta_i \in \mathbb{C}[x, y, z]_{d+2(m-k+1)}$ be the elements such that

$$c_i = \alpha_i t_3 + \beta_i$$

for each $i = 0, \dots, k-1$. Since $-2(y^2 - z^2)(x^2 - z^2) = (w + x^2 + y^2 - z^2)(w - x^2 - y^2 + z^2)$,

$$\frac{t_1 t_2}{t_3} = -\frac{w - x^2 - y^2 + z^2}{2(x+z)(y+z)} = \frac{2(x^2 + y^2 - z^2) - t_3}{2(x+z)(y+z)}. \tag{2.4}$$

Hence we obtain, on the R_X-module $Q(R_X)/R_X$,

$$\begin{aligned} 0 \equiv q \cdot t_1^k t_2^k t_3^{m-k} &= \frac{1}{t_3} \sum_{i=0}^{k-1} (\alpha_i t_3 + \beta_i) t_1^{k-i} t_2^{k-i} \left(x^2 + y^2 - z^2 - 2^{-1} t_3\right)^i \\ &\equiv \frac{1}{t_3} \sum_{i=0}^{k-1} \beta_i t_1^{k-i} t_2^{k-i} (x^2 + y^2 - z^2)^i \\ &\equiv -\frac{w - x^2 - y^2 + z^2}{2(x+z)(y+z)} \sum_{i=0}^{k-1} \beta_i t_1^{k-i-1} t_2^{k-i-1} (x^2 + y^2 - z^2)^i. \end{aligned}$$

Hence we have

$$q' := \sum_{i=0}^{k-1} \beta_i t_1^{k-i-1} t_2^{k-i-1} (x^2 + y^2 - z^2)^i \in (x+z)(y+z)\mathbb{C}[x, y, z].$$

Let $a_i \in \mathbb{C}[x,y,z]$ and $a'_i \in \mathbb{C}[x,y]$ be the polynomials such that $\beta_i = a_i(x+z) + a'_i$. Since the above polynomial is divisible by $x+z$, we have

$$0 \equiv q' \equiv \sum_{i=0}^{k-1} a'_i(2x)^{k-i-1}(x+y)^{k-i-1}y^{2i} \pmod{x+z}.$$

Since $\mathbb{C}[x,y,z]/\langle x+z\rangle \cong \mathbb{C}[x,y]$,

$$\sum_{i=0}^{k-1} a'_i(2x)^{k-i-1}(x+y)^{k-i-1}y^{2i} = 0.$$

Thus there are $g_0,\dots,g_{k-1} \in \mathbb{C}[x,y]$ such that

$$a'_0 = y^2 g_0, \qquad a'_i = y^2 g_i - 2x(x+y)g_{i-1} \quad (i=1,\dots,k-1), \qquad g_{k-1}=0.$$

Since

$$\begin{aligned} & y^2 g_i(t_1t_2)^{k-i-1}(x^2+y^2-z^2)^i + b_{i+1}(t_1t_2)^{k-i-2}(x^2+y^2-z^2)^{i+1} \\ =& \Big(y^2t_1t_2 - 2x(x+y)(x^2+y^2-z^2)\Big)(t_1t_2)^{k-i-2}(x^2+y^2-z^2)^i g_i \\ & \quad + y^2 g_{i+1}(t_1t_2)^{k-i-2}(x^2+y^2-z^2)^{i+1} \\ =& -(x+z)\,G(t_1t_2)^{k-i-2}(x^2+y^2-z^2)^i g_i \\ & \qquad + y^2 g_{i+1}(t_1t_2)^{k-i-2}(x^2+y^2-z^2)^{i+1}, \end{aligned}$$

where $G := 2x^3 + 2x^2y - 2z\,x^2 + 2x\,y^2 - 2xyz + y^3 - y^2z$, we obtain

$$q' = (x+z)\sum_{i=0}^{k-1}(a_i + Gg_i)t_1^{k-i-1}t_2^{k-i-1}(x^2+y^2-z^2)^i.$$

Hence we may assume that $\beta_i \in (x+z)\mathbb{C}[x,y,z]$. Let $\beta'_i \in \mathbb{C}[x,y,z]$ be the polynomial with $\beta_i = (x+z)\beta'_i$ for each $i=0,\dots,k-1$. Then

$$\sum_{i=0}^{k-1} \beta'_i t_1^{k-i-1}t_2^{k-i-1}(x^2+y^2-z^2)^i \in (y+z)\mathbb{C}[x,y,z].$$

By the same argument, we may assume that $\beta'_i \in (y+z)\mathbb{C}[x,y,z]$ for each $i=0,\dots,k-1$. Then there are $b_0,\dots,b_{k-1} \in \mathbb{C}[x,y,z]$ such that $\beta_i =$

$(x+z)(y+z)b_i$ for each i, and

$$\begin{aligned} q &= \frac{1}{t_3^{m-k+1}} \sum_{i=0}^{k-1} (\alpha_i t_3 + b_i(x+z)(y+z)) \frac{(x+z)^i(y+z)^i}{t_3^i} \\ &= \frac{1}{t_3^{m-k}} \left(\sum_{i=0}^{k-1} \alpha_i \frac{(x+z)^i(y+z)^i}{t_3^i} + \sum_{i=0}^{k-1} b_i \frac{(x+z)^{i+1}(y+z)^{i+1}}{t_3^{i+1}} \right). \end{aligned}$$

Therefore the assertion holds. □

Lemma 2.12. *Let $q \in Q(R_X)$ be a homogeneous element of degree 0. If $q \cdot \mathbb{I}_X(nE_1 + mE_2) \subset R_X$, then there are $c_0, \ldots, c_m \in (R_X)_0$ such that*

$$q = \sum_{i=0}^{m} c_i \frac{(x+z)^i(y+z)^i}{t_3^i}.$$

Proof. We prove the assertion by the induction on n. Since

$$\mathbb{I}_X(mE_1 + mE_2) = \langle t_1^{m-i}, t_2^{m-i} t_3^i \mid i = 0, \ldots, m \rangle$$

by Proposition 1.5, it follows from Lemma 2.11 if $n = m$.

Suppose that $n > m$, and $q \cdot \mathbb{I}_X(nE_1 + mE_2) \subset R_X$. Since

$$\mathbb{I}_X(nE_1 + mE_2) = \langle t_3^n \rangle + t_1 \cdot \mathbb{I}_X\big((n-1)E_1 + mE_2\big),$$

there are homogeneous elements $c_0, \ldots, c_m \in (R_X)_1$ such that

$$q = \frac{1}{t_1} \sum_{i=0}^{m} c_i \frac{(x+z)^i(y+z)^i}{t_3^i}$$

by the assumption of the induction. Since $c_i \in (R_X)_1$ for each i, there are $\alpha_i \in \mathbb{C}$ and $\beta_i \in \mathbb{C}[x,y]_1$ such that

$$c_i = \alpha_i t_1 + \beta_i.$$

By $qt_3^n \in R_X$, we have, on the R_X-module $Q(R_X)/R_X$,

$$\begin{aligned} 0 \equiv qt_3^n &= \frac{1}{t_1} \sum_{i=0}^{m} (\alpha_i t_1 + \beta_i) t_3^{n-i} (x+z)^i (y+z)^i \\ &\equiv \frac{w + x^2}{t_1} \sum_{i=0}^{m} \beta_i (2x^2)^{n-i-1} (2y)^i (x+y)^i \pmod{R_X} \end{aligned}$$

since $z \equiv y$ and $(w+x^2)^k \equiv (2x^2)^{k-1}(w+x^2) \pmod{t_1 R_X}$. Hence we obtain

$$\sum_{i=0}^{m} \beta_i (2x^2)^{n-i-1}(2y)^i (x+y)^i = 0$$

as a polynomial of $\mathbb{C}[x,y]$. Then there are $g_0, \dots, g_m \in \mathbb{C}[x,y]$ such that

$$\beta_0 = 2y(x+y)g_0, \qquad \beta_i = 2y(x+y)g_i - 2x^2 g_{i-1} \quad (i = 1, \dots, m), \qquad g_m = 0.$$

Since $\deg \beta_i \leq 1$, we have $g_i = 0$ for each $i = 0, \dots, m$, and $c_i = \alpha_i t_1$. This prove the assertion. □

Proof of Proposition 2.10. Let $q \in \mathbb{C}(X)^\times$ be a rational function with $q \cdot \mathbb{I}_X(nE_1 + mE_2) \subset R_X$. Since

$$\mathbb{I}_X(nE_1 + mE_2) = \left\langle t_1^{n-i} t_2^{m-i} t_3^i,\ t_1^{n-j} t_3^j \;\middle|\; i = 0, \dots, m,\ j = m+1, \dots, n \right\rangle$$

by Proposition 1.5, there are homogeneous elements $c_0, \dots, c_m \in (R_X)_0 = \mathbb{C}$ such that

$$q = \sum_{i=0}^{m} c_i \frac{(x+z)^i (y+z)^i}{t_3^i} \tag{2.5}$$

by Lemma 2.11.

Conversely, suppose that $q \in \mathbb{C}(X)^\times$ is of the form in (2.5). Since

$$t_1^{n-i} t_3^i \cdot \frac{(x+z)^j (y+z)^j}{t_3^j} = t_1^{n-i} t_3^{i-j} (x+z)^j (y+z)^j \in R_X \tag{2.6}$$

for $i = m+1, \dots, n$ and $j = 0, \dots, m$, we have $q \cdot t_1^{n-i} t_3^i \in R_X$ if $i > m$. It is enough to prove that $q \cdot t_1^{n-i} t_2^{m-i} t_3^i \in R_X$ for each $i = 0, \dots, m$. For each $i, j = 0, \dots, m$, put

$$q_{i,j} := t_1^{n-i} t_2^{m-i} t_3^i \cdot \frac{(x+z)^j (y+z)^j}{t_3^j}.$$

If $0 \leq j \leq i \leq m$, then we have

$$q_{i,j} = (x+z)^j (y+z)^j t_1^{n-i} t_2^{m-i} t_3^{i-j} \in R_X. \tag{2.7}$$

If $0 \leq i < j \leq m$, then

$$\begin{aligned} & t_1^{n-i} t_2^{m-i} t_3^i \cdot \frac{(x+z)^j (y+z)^j}{t_3^j} \\ &= \left(\frac{t_1 t_2}{t_3}\right)^{j-i} t_1^{n-j} t_2^{m-j} (x+z)^j (y+z)^j \\ &= \left(-\frac{w - x^2 - y^2 + z^2}{2(x+z)(y+z)}\right)^{j-i} t_1^{n-j} t_2^{m-j} (x+z)^j (y+z)^j \\ &= \left(-\frac{1}{2}\right)^{j-i} t_1^{n-j} t_2^{m-j} (x+z)^i (y+z)^i (w - x^2 - y^2 + z^2)^{j-i} \in R_X \end{aligned} \tag{2.8}$$

by (2.4). □

We prove Theorem 1.6 in the case of $n \geq m$.

Proof of Theorem 1.6. Let D be an effective divisor on X with $D \sim nE_1 + mE_2$. By Lemma 2.2, there is $q \in \mathbb{C}(X)^\times$ such that

$$q \cdot \mathbb{I}_X(nE_1 + mE_2) = \mathbb{I}_X(D) \subset R_X.$$

By Proposition 2.10, there are $c_0, \ldots, c_m \in \mathbb{C}$ such that

$$q = \sum_{j=0}^{m} c_j \frac{(x+z)^j (y+z)^j}{t_3^j}.$$

By (2.6), (2.7) and (2.8), $\mathbb{I}_X(D) \subset R_X$ is generated by

$$\sum_{j=0}^{i} c_j A_{i,j} + \sum_{j=i+1}^{m} (-2)^{i-j} c_j B_{i,j} \quad \text{and} \quad \sum_{j=m+1}^{n} c_j A'_{i,j}$$

for $i = 0, \ldots, m$. □

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