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# System of Axioms of Euclidean Geometry

By

Yoshifumi Ito

*Professor Emeritus, Tokushima University*
*209-15 Kamifukuman Hachiman-cho*
*Tokushima 770-8073, JAPAN*
*e-mail address* : *itoyoshifumi@fd5.so-net.ne.jp*

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**Abstract**

In this paper, we define Euclidean geometry and prove the existence theorem of Euclidean geometry by using the axiomatic method.

Thereby we give the system of Axioms of Euclidean Geometry and prove its consistency.

Thus we give the complete solution of the problem of the foundation of Euclidean geometry.

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## Introduction

In this paper, we give the true expression of Euclid's system of axioms. Thereby we define Euclidean geometry and prove the existence theorem of Euclidean geometry.

Euclidean geometry was originated by Euclid's "Principle". In Euclid's system of axioms, the concepts of the point, the line and the plane were not defined clearly.

Therefore there was the problem of its truth. Hilbert gave the regularization of Euclidean geometry by using Hilbert's system of axioms in his "foundation of geometry".

Before Cantor's set theory, any one does not know the mathematical concepts of the point, the line and the plane.

Euclid's system of axioms and Hilbert's system of axioms do not give the concrete plane geometry. They are only the abstract theory.

Namely, in these systems of axioms, they could not define concretely the concepts of the point, the line and the plane.

By using the Cantor's set theory or the exact set theory, we can define the Euclidean plane, and the line and the point there.

Therefore, by defining the Euclidean plane defined by using Weyl's system of axioms, we can complete the Euclid's "Principle".

At last, we express my heartfelt thanks to my wife Mutuko for her cooperation of making this file of the manuscript.

# 1 Proof of the consistency of the system of axioms of Euclidean geometry

In this section, we define Euclidean geometry and prove its existence theorem by constructing its model. Thereby we can prove the consistency of the system of axioms of Euclidean geometry. As for these facts, we refer to Ito [8], "Foundation of Linear algebra", Chapter 16.

**Definition 1.1(Euclidean geometry)** Euclidean geometry is the geometry of the figures constructed by points and lines in the plane, which satisfies the axioms (1)$\sim$(5) in the following:

(1) We can draw the line from an arbitrary point to another arbitrary point.

(2) We can extend the finite line (the line segment) to a straight line continuously.

(3) We can draw a circle with an arbitrary point and a distance (the radius).

(4) All right angles are mutually equal.

(5) If one straight line intersects two straight lines and the sum of the inner angles on the same side is smaller than the 2-times right angles, the two straight lines intersect on the side where the sum of the inner angles is smaller than the 2-times right angles when these two straight lines are belonged infinitely.

**Remark 1.1** In Definition 1.1, Euclidean geometry is the geometry of the planar figures drawn by using the ruler and the compass on the plane.

Next, we prove the existence theorem of Euclidean geometry. For that purpose, we prove this by constructing a model of Euclidean geometry.

Thereby we can prove the consistency of the system of axioms of Euclidean geometry.

**Definition 1.2 (Plane)** We assume that the set of points S is not the empty set. We define that the combined concept $(S,\ V^2)$ of the set of points S and the 2-dimensional real metric vector space $V^2$ is the Euclidean plane if we have the conditions (i), (ii) in the following:

(i) For an arbitrary pair $A$ and $B$ of points in $S$, we have only one vector $\boldsymbol{a}$ in $V^2$. We express this as $\boldsymbol{a} = (\overrightarrow{AB})$. Further, for an arbitrary point $A$ in $S$ and an arbitrary vector $\boldsymbol{a}$ in $V^2$, we have only one point $B$ in S so that we have the equality $(\overrightarrow{AB}) = \boldsymbol{a}$.

(ii) If we have two equalities $\boldsymbol{a} = (\overrightarrow{AB})$ and $\boldsymbol{b} = (\overrightarrow{BC})$, we have the equality $\boldsymbol{a} + \boldsymbol{b} = (\overrightarrow{AC})$.

Then we denote $\boldsymbol{E}^2 = (S,\ V^2)$. We happen to say simply that the Euclidean plane is the plane.

We say that this is Weyl's system of axioms.

**Definition 1.3 (Point and vector)** Assume that $\boldsymbol{E}^2 = (S,\ V^2)$ is the Euclidean plane. We say that a point in $S$ is a **point** in $\boldsymbol{E}^2$ and a **vector** in $V^2$ is a **vector** in $\boldsymbol{E}^2$.

**Definition 1.4 (Origin and position vector)** We fix one point $O$ in $\boldsymbol{E}^2$ and we say that this point is the **origin.** Then, for an arbitrary point $P$ in $\boldsymbol{E}^2$, we say that the vector $\boldsymbol{a} = (\overrightarrow{OP})$ is the **position vector** of the point $P$

**Definition 1.5 (Line segment)** Assume that $\boldsymbol{a}_0$ and $\boldsymbol{a}$ are two vectors in $\boldsymbol{E}^2$ and we have $\boldsymbol{a} \neq \boldsymbol{0}$. Now we assume that two points $A$ and $B$ are the points of position vectors $\boldsymbol{a}$ and $\boldsymbol{a}_0 + \boldsymbol{a}$ respectively. Then we define that the line segment connecting the points $A$ and $B$ is the set of all points whose position vectors are the vectors $\boldsymbol{a}_0 + t\boldsymbol{a}$, $(0 \leq t \leq 1)$. Then we denote this line segment as $AB$. We define that the length of the line segment $AB$ is $\overline{AB} = \|\boldsymbol{a}\|$. Here $\|\boldsymbol{a}\|$ denotes the norm of $\boldsymbol{a}$.

**Definition 1.6 (Straight line)** Assume that $\boldsymbol{a} = (\overrightarrow{OA})$ is a position vector of a point $A$ in $\boldsymbol{E}^2$ and we assume that the vector $\boldsymbol{a}$ in $\boldsymbol{E}^2$ is not equal to $\boldsymbol{0}$. Then we define that the set of all points $P$ whose position vectors are $\boldsymbol{a}_0 + t\boldsymbol{a}$, $(-\infty < t < \infty)$ is a straight line with the direction vector $\boldsymbol{a}$.

**Definition 1.7 (Triangle)** We denote the points whose position vectors are three vectors $\boldsymbol{a}_0,\ \boldsymbol{a}_0+\boldsymbol{a}_1$ and $\boldsymbol{a}_0+\boldsymbol{a}_2$ in $\boldsymbol{E}^2$ as $A,\ B$ and $C$ respectively. Then we say that the set of all points whose position vectors are the vectors $\boldsymbol{a}_0+t_1\boldsymbol{a}_1+t_2\boldsymbol{a}_2,\ (0 \leq t_1,\ t_2 \leq 1,\ 0 \leq t_1+t_2 \leq 1)$ is a 2-dimensional triangle $ABC$. Further we say that the set of all points whose position vectors are the vectors $\boldsymbol{a}_0+t_1\boldsymbol{a}_1+t_2\boldsymbol{a}_2,\ \Big( (t_1,\ t_2) \in ([0,\ 1]\times\{0\}\bigcup\{0\}\times[0,\ 1]\bigcup\{(t_1,\ t_2);\ t_1 \geq 0,\ t_2 \geq 0,\ t_1+t_2=1\}\ )\Big)$ is a triangle $ABC$.

**Definition 1.8 (Parallel)** We say that two straight lines $\alpha_1$ and $\alpha_2$ are **parallel** if $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are linearly dependent when we assume that the direction vectors of $\alpha_1$ and $\alpha_2$ are $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ respectively.

**Definition 1.9 (Length)** For two points $A$ and $B$ in $\boldsymbol{E}^2$, the length $\overline{AB}$ of a directed line segment $\overrightarrow{AB}$ is defined to be the norm $\|\boldsymbol{a}\|$ of the vector $(\overrightarrow{AB})=\boldsymbol{a}$. We also say that this is the **length** of the line segment $AB$.

**Definition 1.10 (Angle of two line segments)** The angle $\alpha$ of two line segments is defined to be

$$\alpha = \cos^{-1}(\boldsymbol{a},\ \boldsymbol{b})$$

when we assume that the direction vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ of these line segments are not equal to the zero vector $\boldsymbol{0}$. Here $(\boldsymbol{a},\ \boldsymbol{b})$ denotes the inner product of $\boldsymbol{a}$ and $\boldsymbol{b}$.

By using Definition 1.2 $\sim$ Definition 1.10, we prove the existence of a model of Euclidean geometry of the plane in Theorem 1.1.

**Theorem 1.1 (Existence of a model of Euclidean geometry)** *On the Euclidean plane* $\boldsymbol{E}^2=(S,\ V^2)$, *we have the statements* (1)$\sim$(5) *in the following*:

(1) *We can draw a line from an arbitrary point to another arbitrary point.*

(2) *We can extend continuously the finite line (the line segment) to one straight line.*

(3) *We can draw a circle with an arbitrary point and a distance (the radius).*

(4) *All right angles are mutually equal.*

(5) *If one straight line intersects two straight lines and the sum of two inner angles on the same side is less than the* 2*-times right angles, then these two straight lines intersect on the side where the sum of two inner angles is less than the* 2*-times right angles when these two straight lines are belonged infinitely.*

This is a model of Euclidean geometry.

Because we proved the existence of a model of Euclidean geometry by the Theorem 1.1, we proved the consistency of the system of axioms of Euclidean geometry.

Thereby we prove that the geometry in Euclid's principle holds in the Euclidean plane $\boldsymbol{E}^2 = (S,\ V^2)$.

# 2 Study on the system of axioms of Euclidean geometry

In this section, we study the system of axioms of Euclidean geometry.

As for the results in this section, we refer to Ito "Fushigi no Izumi(mathematics I)", Chapter 14.

What are the mathematical concepts such as the set, the number and Euclidean geometry.

By virtue of the axiomatic method used until now, the system of axioms is concerned as the final base point of the logical thinking in order to prove the derived theorems by the system of axioms. So that, the system of axioms does not define the concept of numbers as an ideal existence. The proof of the theorems of the mathematical concept is the biggest purpose and we consider the system of axioms as its base point. Then the mathematical concepts, for examples, the point, the line and the plane in Euclidean geometry are expressed simply as the undefined terminologies, and the system of axioms only defines their relations.

By virtue of my new axiomatic method, the system of axioms is the condition for the definition of the mathematical concepts and the consistency of the system of axioms is proved by virtue of the proof of the existence theorem of such a mathematical concept. All mathematical concepts are the conceptual existences in the set theory I.

A $d$-dimensional metric affine space is a certain type of manifold and it is a flat manifold. A $d$-dimensional metric vector space is attached at each point of the manifold.

These $d$-dimensional metric vector spaces are all isomorphic, and the $d$-dimensional metric vector space at each point is put on each other by the parallel translation.

Then the $d$-dimensional metric vector space at each point plays the role of certain type of the coordinate neighborhood. This is the difference between a $d$-dimensional metric vector space as itself and a $d$-dimensional metric affine space.

Thereby, the differential and integral calculus on the $d$-dimensional metric affine space is realized as the differential and integral calculus on a manifold.

The differential and integral calculus is principally realized on a $d$-dimensional Euclidean space as a $d$-dimensional metric vector space.

In the expression of the system of axioms which Euclid gave in his Principle, the definitions of the various types of the mathematical concepts are not given exactly. The point, the line and the plane etc. are considered as the "undefined terminology" and their relations are defined as the system of axioms. Therefore, this Principle is considered to be a systematical formulation of the experienced knowledges of the geometry at that time. If we consider that as a first book of the mathematical results at that time, this consideration is considered as a natural fact.

After this, the definitions of the concepts of the natural numbers and the real numbers were the systematical formulation of the experienced knowledges about the natural numbers and the real numbers. They are not the mathematically exact definitions. Recently I complete the definitions of the natural numbers and the real numbers and the proofs of their existence theorems by using my new axiomatic method.

In the definition of the system of axioms of Euclidean geometry given by Euclid himself, Euclid defines the point, the straight line and the plane as the undefined terminology and prescribe their relations as the system of axioms. In Hilbert's system of axioms of Euclidean geometry, he defined the point, the straight line and the plane as the undetermined terminology and prescribed their relations as the system of axioms. At this point, Hilbert's system of axioms of Euclidean geometry is essentially the same thing as Euclid's system of axioms.

In this time, we consider that the plane is the 2-dimensional metric affine space in the set theory I. There, the point, the straight line are defined correctly as the sets of points in the plane. Then we adapt the Euclid's system of axioms as the relations satisfied by the point, the straight line and the plane. Thereby the point, the straight line and the plane are defined as the natural existence of the set theory I and they are defined without any ambiguity. Euclidean geometry has the natural foundation as the existence in the set theory I.

Euclidean plane, namely the 2-dimensional metric affine space and Euclidean geometry are not identical. Euclidean geometry is the mathematics of the figures on the plane. On the Euclidean plane, we can study not only Euclidean geometry but also the differential and integral calculus and the theories of physical phenomena.

The exact definition of the concepts of the point, the straight line and the plane is not only the problem of Euclidean geometry but also the problem of the exact definition of Euclidean plane and the exact determination of its structure. The problem of Euclidean geometry is the study of the properties of the figures drawn by using the ruler and compass on the Euclidean plane.

When we study the Euclidean geometry on the plane, we consider the planar figures which we can draw by using the ruler and the compass. The experienced knowledge is that these planar figures are the figures composed of the points and the line segments in the plane. We have not seen the figures drawn by using the chair, the desk and the beer jokey as Hilbert's said.

This is the experienced knowledge. When we study them logically, we study them by formulating these experienced knowledges systematically by using the system of axioms. The nonsense idea of Hilbert's consideration does not appear. Anywhere, This means the biggest mistakes because he formalize the mathematics as the system of only the nonsense symbolic calculation.

If we consider the extremely abstract formalization, the considered fact becomes nonsense.

What form is the "Triangle" drawn by using the chair, the desk and the beer jokey as Hilbert said.

Since the word does not express any meaning, we cannot understand the meaning of that word. Why we can study the mathematics by using Hilbert's idea. Even if we can understand the Euclidean geometry by using the meaningless symbols and words as the undetermined terminology said by Euclid and Hilbert, it does not mean that we understand the geometry purely theoretically.

In any way, he consider by himself that he understand the properties of the figures by axiomatic method on the basis of the accumulation of the experienced knowledges.

Thus, the understanding of the expression by using the meaningless symbols and words is considered as the misunderstanding by virtue of those considerations.

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