On Some Formulas for \( \pi/2 \)

By

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Abstract

In his paper [1], J.G. Goggins has shown a formula which relates \( \pi \) and Fibonacci numbers. In our paper [2], we have proved a generalized version of this formula. In this note, we shall prove formulas which generalize Fibonacci number to certain binary recurrence sequences.

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Introduction

In [1], J.G. Goggins has shown the following simple but very interesting formula

\[
\frac{\pi}{4} = \sum_{n=1}^{\infty} \tan^{-1}(1/F_{2n+1}),
\]  \hspace{1cm} (1)

where \( F_n \) is the \( n \)th Fibonacci number. We note this formula is also given as the formula (f) in the text [5] chapter 3. Since \( F_1 = 1 \), we see \( \frac{\pi}{4} = \tan^{-1}(1/F_1) \). Thus (1) is equivalent to the following formula

\[
\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(1/F_{2n+1}).
\]  \hspace{1cm} (2)

The purpose of this short note is to generalize this formula on Fibonacci number to two formulas on binary recurrence sequences, that is, to the following
two formulas

$$\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(t/u_{2n+1}),$$

(3)

$$\frac{\pi}{2} = \sum_{n=-\infty}^{\infty} \tan^{-1}(t/v_{2n}).$$

(4)

Here \( \{u_n\} \) is the Lucas sequences associated to the parameter \((t, -1)\) and \( \{v_n\} \) is the companion Lucas sequences associated to the parameter \((t, -1)\), respectively.

First of all, let us recall the fundamental properties of \( u_n \) and \( v_n \). Let \( t \) be a positive integer and \( \{u_n\} \) and \( \{v_n\} \) be the binary recurrence sequences defined by putting

$$\begin{align*}
\begin{cases}
  u_{n+2} = tu_{n+1} + u_n, \\
  v_{n+2} = tv_{n+1} + v_n,
\end{cases}
\end{align*}$$

with initial terms \( u_0 = 0, u_1 = 1 \) and \( v_0 = 2, v_1 = t \).

Put \( \varepsilon = (t + \sqrt{t^2 + 4})/2 \) and \( \bar{\varepsilon} = (t - \sqrt{t^2 + 4})/2 \). Then one knows the following Binet’s formula

$$\begin{align*}
\begin{cases}
  u_n = (\varepsilon^n - \bar{\varepsilon}^n)/\sqrt{t^2 + 4}, \\
  v_n = \varepsilon^n + \bar{\varepsilon}^n,
\end{cases}
\end{align*}$$

Put \( \alpha_{2n} = \tan^{-1}(1/u_{2n}) \) and \( \alpha_{2n-1} = \tan^{-1}(t/u_{2n-1}) \) for any positive index \( n \). Then we can show the following proposition.

**Proposition 1.** For any integer \( n \geq 1 \), \( \alpha_{2n} = \alpha_{2n+1} + \alpha_{2n+2} \).

Proof. We have

$$\tan(\alpha_{2n+1} + \alpha_{2n+2}) = \frac{t/u_{2n+1} + 1/u_{2n+2}}{1 - t/(u_{2n+1}u_{2n+2})} = \frac{tu_{2n+2} + u_{2n+1}}{u_{2n+1}u_{2n+2} - t}$$

By virtue of the Binet’s formula, we see

$$u_{2n+1}u_{2n+2} - t = (\varepsilon^{2n+1} - \bar{\varepsilon}^{2n+1})(\varepsilon^{2n+2} - \bar{\varepsilon}^{2n+2})/(t^2 + 4) - t$$

$$= (\varepsilon^{4n+3} + \bar{\varepsilon}^{4n+3} + \varepsilon + \bar{\varepsilon})/(t^2 + 4) - t = (\varepsilon^{4n+3} + \bar{\varepsilon}^{4n+3} - t^3 - 3t)/(t^2 + 4).$$
On the other hand, we also have

\[
\begin{align*}
    u_{2n} u_{2n+3} &= (\varepsilon^{2n} - \varepsilon^{2n}) (\varepsilon^{2n+3} - \varepsilon^{2n+3}) / (t^2 + 4) \\
    &= (\varepsilon^{4n+3} + \varepsilon^{4n+3} - \varepsilon^3 - \varepsilon^3) / (t^2 + 4) = (\varepsilon^{4n+3} + \varepsilon^{4n+3} - t^3 - 3t) / (t^2 + 4).
\end{align*}
\]

Thus we have shown

\[
\tan(\alpha_{2n+1} + \alpha_{2n+2}) = \frac{1}{u_{2n}} = \tan(\alpha_{2n}),
\]

which completes the proof.

From this proposition, we have \(\alpha_{2n} - \alpha_{2n+2} = \alpha_{2n+1}\) for any \(n \geq 1\). Then we have

\[
\sum_{n=1}^{\infty} \tan^{-1}(t/u_{2n+1}) = \sum_{n=1}^{\infty} \alpha_{2n+1} = \sum_{n=1}^{\infty} (\alpha_{2n} - \alpha_{2n+2}) = (\alpha_2 - \alpha_4) + (\alpha_4 - \alpha_6) + \cdots + (\alpha_{2n} - \alpha_{2n+2}) + \cdots = \alpha_2.
\]

Since \(\alpha_2 = \tan^{-1}(1/t) = \frac{\pi}{2} - \tan^{-1}(t/u_1)\), we have shown the formula (3).

Now we shall show the formula (4) similarly. Put \(\beta_{2n} = \tan^{-1}(t/v_{2n})\) and \(\beta_{2n-1} = \tan^{-1}(2/v_{2n-1})\) for any positive index \(n\). Then we can show the following proposition.

**Proposition 2.** For any integer \(n \geq 1\), \(2\beta_{2n} = \beta_{2n-1} - \beta_{2n+1}\).

Proof. We have

\[
\tan(\beta_{2n-1} - \beta_{2n+1}) = \frac{2/v_{2n-1} - 2/v_{2n+1}}{1 + 4/(v_{2n-1}v_{2n+1})} = \frac{2(v_{2n+1} - v_{2n-1})}{v_{2n-1}v_{2n+1} + 4}.
\]

By virtue of the Binet’s formula, we see

\[
\begin{align*}
    v_{2n-1}v_{2n+1} + 4 &= (\varepsilon^{2n-1} + \varepsilon^{2n+1})(\varepsilon^{2n-1} + \varepsilon^{2n-1}) + 4 \\
    &= (\varepsilon^{4n} + \varepsilon^{4n}) - (\varepsilon^2 + \varepsilon^2) + 4 = (\varepsilon^{4n} + \varepsilon^{4n}) - (t^2 + 2) + 4 \\
    &= (\varepsilon^{2n} + \varepsilon^{2n})^2 - t^2 = v_{2n}^2 - t^2.
\end{align*}
\]
On the other hand, we have
\[
\tan(2\beta_{2n}) = \frac{t/v_{2n} + t/v_{2n}}{1 - (t/v_{2n})^2} = \frac{2tv_{2n}}{v_{2n}^2 - t^2}.
\]

Thus we have shown
\[
\tan(\beta_{2n-1} - \beta_{2n+1}) = \tan(2\beta_{2n}),
\]
which completes the proof.

From this proposition, we have \(\beta_{2n-1} - \beta_{2n+1} = 2\beta_{2n}\) for any \(n \geq 1\).

Then we have
\[
\sum_{n=1}^{\infty} 2\tan^{-1}(t/v_{2n}) = \sum_{n=1}^{\infty} 2\beta_{2n} = \sum_{n=1}^{\infty} (\beta_{2n-1} - \beta_{2n+1}) = \beta_1 = \tan^{-1}(2/t).
\]

Since \(v_{-2n} = v_{2n}\), one knows that \(\tan^{-1}(t/v_{-2n}) = \tan^{-1}(t/v_{2n})\).

Hence we have
\[
\sum_{n=-\infty}^{\infty} \tan^{-1}(t/v_{2n}) = 2 \left( \sum_{n=1}^{\infty} \tan^{-1}(t/v_{2n}) \right) + \tan^{-1}(t/v_0)
\]
\[= \tan^{-1}(2/t) + \tan^{-1}(t/2) = \frac{\pi}{2},\]
which completes the proof of (4).

Now we have completely proved two formulas of (4), which we shall state as the following theorem.

**Theorem.** With the above notations, we have the following formulas,

\[
\frac{\pi}{2} = \sum_{n=0}^{\infty} \tan^{-1}(t/u_{2n+1}),
\]
\[= \sum_{n=-\infty}^{\infty} \tan^{-1}(t/v_{2n}).\]
References


