Existence of Global and Bounded Solutions for Damped Sublinear Wave Equations

By

Kosuke Ono *

Department of Mathematical and Natural Sciences
The University of Tokushima
Tokushima 770-8502, JAPAN

e-mail: ono@ias.tokushima-u.ac.jp

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Abstract

We study the initial–boundary value problem for the sublinear wave equations with a linear damping: $u'' - \Delta u - \omega \Delta u' + \delta u' = \gamma |u|^{p-2}u$ with the homogeneous Dirichlet boundary condition and $H^1_0(\Omega) \times L^2(\Omega)$-data condition under $\omega \geq 0$ and $\delta > -\omega \lambda_1$. When $1 < p < 2$, we show that the (local) weak solutions are global and uniformly bounded in time $t \geq 0$.

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1 Introduction

We consider the initial-boundary value problem for the following semilinear wave equation:

$$u'' - \Delta u - \omega \Delta u' + \delta u' = f(u), \quad u = u(x,t), \quad \text{in } \Omega \times [0, \infty)$$

(1)

with homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial \Omega$$

and initial conditions

$$u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x),$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $' = \partial/\partial t$, $\Delta = \nabla \cdot \nabla = \sum_{j=1}^N \partial^2/\partial x_j^2$ is Laplacian, $\omega$ and $\delta$ are constants such that $\omega \geq 0$

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and \( \delta > -\omega \lambda_1 \) with \( \lambda_1 \) being the first eigenvalue of the operator \(-\Delta\) under the homogeneous Dirichlet boundary condition, that is,

\[
\lambda_1 = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\|\nabla u\|^2}{\|u\|^2},
\]

and

\[
f(u) = \gamma|u|^{p-2}u, \quad \gamma > 0, \quad p > 1.
\]

In the supperlinear case \( p > 2 \), it is well known that the so-called potential well method is useful to the analysis of global existence for problem (1). (see Sattinger [21], Tsutsumi [23], Payne-Sattinger [20], and also, [8], [9], [14], [18]), and moreover, the concavity method is applied to the analysis of finite time blow-up phenomena (see Tsutsumi [23], Levine [10], [11], and also, [1], [2], [3], [7], [13], [15], [16], [22]).

In order to explain some known results for \( p > 2 \), we define the total energy associated with (1) by

\[
E(u, u') = \frac{1}{2}\|u'\|^2 + J(u)
\]

where we put

\[
J(u) = \frac{1}{2}\|\nabla u\|^2 - \frac{\gamma}{p}\|u\|_p^p
\]

\[
= \left( \frac{1}{2} - \frac{1}{p} \right)\|\nabla u\|^2 + \frac{1}{p}I(u)
\]

with

\[
I(u) = \|\nabla u\|^2 - \gamma\|u\|_p^p,
\]

and we define the mountain pass level \( d \) (also known as the potential well depth) by

\[
d = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \left( \sup_{\lambda \geq 0} J(\lambda u) \right)
\]

(see Sattinger [21], Tsutsumi [23], Payne-Sattinger [20]).

When the power \( p \) in (2) satisfies that \( p > 2 \) and \( p \leq 2(N-1)/(N-2) \) if \( N \geq 3 \), many authors have already studied on global existence or finite time blow-up of (local) weak solutions in the class \( C([0, T); H^1_0(\Omega)) \cap C^1([0, T); L^2(\Omega)) \) for the problem (1) with the initial data \( (u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega) \) satisfying suitable conditions: (i) if \( E(u_0, u_1) < d \) and \( I(u_0) > 0 \), then there exists a unique global solution \( u(t) \) satisfying \( \|u(t)\|_{H^1} + \|u'(t)\| \to 0 \) as \( t \to \infty \); (ii) if \( E(u_0, u_1) < d \) and \( I(u_0) < 0 \), then the local solution \( u(t) \) blows up at some finite time, that
is, there exists a finite time $T^* < \infty$ such that $\|u(t)\|_{H^1} \to \infty$ as $t \to T^*$; moreover (iii) when $\omega = 0$, if $E(u_0, u_1) \geq d$, $I(u_0) < 0$, $\|u_0\| \geq \sup\{\|\phi\|_H \phi \in H^1_0(\Omega) \wedge \{0\}\}$ with $(1/2 - 1/p)\|\nabla \phi\|^2 \leq E(u_0, u_1)$, and $\int_0^t u_0 u_1 \, dx \geq 0$, then the local solution $u(t)$ blows up at some finite time (see Gazzola-Squassina [5]). We note that if $E(u(t), u'(t)) \geq d$ for all $t \geq 0$, then $\lim_{t \to \infty} E(u(t), u'(t))$ exists, and when $\omega = 0$, $p > 2$, and $p < 2(N - 1)/(N - 2)$ if $N \geq 3$ or $p \leq 6$ if $N = 2$, the local solution $u(t)$ is global and bounded (see Esquivel-Avila [4]).

On the other hand, when the power $p$ in (2) satisfies that $1 < p \leq 2$, we see that for the initial data $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$, the (local) weak solution $u(t)$ in the class $C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ is global. In particular, from Remark 3.10 in Gazzola and Squassina [5], we know that the global solution $u(t)$ satisfies

$$\|u(t)\|_{H^1} + \|u'(t)\| \leq C(1 + t)^{p/(4-2p)} \quad \text{if } p < 2$$

$$\|u(t)\|_{H^1} + \|u'(t)\| \leq Ce^{\alpha t} \quad \text{if } p = 2$$

with some $\alpha > 0$, for $t \geq 0$, but we can not know boundedness of global solutions.

The purpose of this paper is to show boundedness of global solutions of (1) in the case $1 < p < 2$ (i.e. sublinear case).

Our main result is as follows.

**Theorem 1.1** Let $1 < p < 2$, and let $\omega \geq 0$ and $\delta > -\omega \lambda_1$. Suppose that the initial data $(u_0, u_1)$ belong to $H^1_0(\Omega) \times L^2(\Omega)$. Then, the problem (1) admits a unique global solution $u(t)$ in the class $C([0, \infty); H^1_0(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$ satisfying

$$\|u(t)\|_{H^1} + \|u'(t)\| \leq C + CI_0e^{-kt}$$

with some constants $C > 0$, $\tilde{k} > 0$, and $I_0 = \|\nabla u_0\| + \|u_1\|$, for $t \geq 0$.

On the other hand, in the case $p = 2$ we have the following.

**Theorem 1.2** Let $p = 2$, and let $\omega \geq 0$ and $\delta > -\omega \lambda_1$. Suppose that the initial data $(u_0, u_1)$ belong to $H^1_0(\Omega) \times L^2(\Omega)$. Then, the problem (1) admits a unique global solution $u(t)$ in the class $C([0, \infty); H^1_0(\Omega)) \cap C^1([0, \infty); L^2(\Omega))$ satisfying that $\|u(t)\|_{H^1} + \|u'(t)\| \leq CI_0e^{\alpha t}$, and moreover, if $\gamma < \lambda_1$,.

$$\|u(t)\|_{H^1} + \|u'(t)\| \leq CI_0e^{-\tilde{k}t}$$

with some constants $C > 0$, $\tilde{\alpha} > 0$, $\tilde{k} > 0$, and $I_0 = \|\nabla u_0\| + \|u_1\|$, for $t \geq 0$.

We use only familiar functional spaces and omit the definitions. We denote $L^p(\Omega)$-norm by $\|\cdot\|_p$ (we often write $\|\cdot\| = \|\cdot\|_2$ for simplicity). Positive constants will be denoted by $C$ and will change from line to line.
2 Proofs

By applying the Banach contraction mapping theorem, we obtain the following local existence theorem (e.g. see [6], [12], [17], [19]).

**Proposition 2.1** Let $p > 1$ and $p \leq 2(N - 1)/(N - 2)$ if $N \geq 3$. Suppose that the initial data $(u_0, u_1)$ belong to $H^1_0(\Omega) \times L^2(\Omega)$. Then, there exists a unique (local) weak solution $u(t)$ in the class $C([0,T); H^1_0(\Omega)) \cap C^1([0,T); L^2(\Omega))$ of problem (1), that is,

\[
\frac{d}{dt} \int_{\Omega} u'(t) w \, dx + \int_{\Omega} \nabla u(t) \nabla w \, dx + \omega \int_{\Omega} \nabla u'(t) \nabla w \, dx + \delta \int_{\Omega} u'(t) w \, dx = \int_{\Omega} f(u(t)) w \, dx
\]

a.e. in $(0,T)$ for every $w \in H^1_0(\Omega)$.

Moreover, if $\sup_{0 \leq t \leq T} \left( ||u(t)||_{H^1} + ||u'(t)|| \right) < \infty$, then the solution $u(t)$ can be continued to $T + \epsilon$ for some $\epsilon > 0$.

**Proof of Theorem 1.1.** Multiplying (1) by $u'$ and integrating it over $\Omega$, we have

\[
\frac{d}{dt} E_1(t) + \omega ||\nabla u'(t)||^2 + \delta ||u'(t)||^2 = \int_{\Omega} f(u(t))u'(t) \, dx ,
\]

where $E_1(t)$ is defined by

\[
E_1(t) = E_1(u(t), u'(t)) = \frac{1}{2} \left( ||u'(t)||^2 + ||\nabla u(t)||^2 \right) .
\]

And, multiplying (1) by $u$ and integrating it over $\Omega$, we have

\[
\frac{d}{dt} \frac{1}{2} \left( \omega ||\nabla u(t)||^2 + \delta ||u(t)||^2 + 2 \int_{\Omega} u(t)u'(t) \, dx \right) + ||\nabla u(t)||^2 - ||u'(t)||^2 = \int_{\Omega} f(u(t))u(t) \, dx .
\]

Then, taking (3) $+ \epsilon \times (4)$ for any small $\epsilon > 0$, we have

\[
\frac{d}{dt} F_1(t) + G_1(t) = \int_{\Omega} f(u(t)) (u'(t) + \epsilon u(t)) \, dx ,
\]

where $F_1(t)$ and $G_1(t)$ are defined by

\[
F_1(t) = E_1(t) + \frac{\epsilon}{2} \left( \omega ||\nabla u(t)||^2 + \delta ||u(t)||^2 + 2 \int_{\Omega} u(t)u'(t) \, dx \right)
\]

\[
G_1(t) = \frac{\epsilon}{2} \left( \omega ||\nabla u(t)||^2 + \delta ||u(t)||^2 + 2 \int_{\Omega} u(t)u'(t) \, dx \right) .
\]
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\[ G_1(t) = (\omega ||\nabla u'(t)||^2 + \delta ||u'(t)||^2) + \varepsilon \left(||\nabla u(t)||^2 - ||u'(t)||^2\right). \]

Here, it is easy to see from the Cauchy inequality and the Poincaré inequality that

\[ F_1(t) \leq C(||u'(t)||^2 + ||\nabla u(t)||^2) \tag{6} \]

and

\[ G_1(t) \geq (\delta + \omega \lambda_1 - \varepsilon)||u'(t)||^2 + \varepsilon||\nabla u(t)||^2. \tag{7} \]

Thus, if \( \delta + \omega \lambda_1 > 0 \), choosing small \( \varepsilon > 0 \), we have from (5)–(7) that

\[ \frac{d}{dt} F_1(t) + 2kF_1(t) \leq \int_{\Omega} f(u(t)) (u'(t) + \varepsilon u(t)) \, dx \tag{8} \]

with some constant \( k > 0 \). Moreover, we observe from the Young inequality and the Poincaré inequality with \( p < 2 \) that

\[ \int_{\Omega} f(u(t)) (u'(t) + \varepsilon u(t)) \, dx \leq C||u'(t)||_p ||u(t)||_{p-1}^p + C||u(t)||_p^p \]

\[ \leq C||u'(t)|| ||\nabla u(t)||_{p-1}^p + C||\nabla u(t)||_p^p \tag{9} \]

and by \( \delta + \omega \lambda_1 > 0 \),

\[ F_1(t) \geq F_1(t) + \frac{\varepsilon}{2} \left((\delta + \omega \lambda_1)||u(t)||^2 - 2||u(t)||||u'(t)||\right) \]

\[ \geq \frac{1}{2}(1 - C\varepsilon)||u'(t)||^2 + \frac{1}{2}||\nabla u(t)||^2. \tag{10} \]

Thus, choosing small \( \varepsilon > 0 \), we have from (8)–(10) that

\[ \frac{d}{dt} F_1(t) + 2kF_1(t) \leq CF_1(t)^{p/2} \leq kF_1(t) + C., \]

where we used the Young inequality together with the fact that \( 1/2 < p/2 < 1 \) at the last inequality, and hence,

\[ F_1(t) \leq \frac{C}{k} + F_1(0)e^{-kt}. \tag{11} \]

Therefore, we obtain from (10) and (11) that

\[ ||u'(t)||^2 + ||\nabla u(t)||^2 \leq CF_1(t) \leq C + CF_0^2 e^{-kt} \]

for \( t \geq 0. \)
Proof of Theorem 1.2. Since \( p = 2 \) in (2), we have from (3) and the Poincaré inequality that
\[
\frac{d}{dt} E_1(t) \leq \gamma \|u(t)\| \|u'(t)\| \leq \alpha E_1(t)
\]
with some constant \( \alpha > 0 \), and hence, we have
\[
\|u'(t)\|^2 + \|\nabla u(t)\|^2 \leq 2E_1(0)e^{\alpha t}
\]
for \( t \geq 0 \).

Next, let \( \gamma < \lambda_1 \). Since \( p = 2 \) in (2), we have from (3) and (4) that
\[
\frac{d}{dt} E(t) + \omega \|\nabla u'(t)\|^2 + \delta \|u'(t)\|^2 = 0
\]
(12)
and
\[
\frac{d}{dt} \frac{1}{2} \left( \omega \|\nabla u(t)\|^2 + \delta \|u(t)\|^2 + 2 \int_{\Omega} u(t)u'(t) \, dx \right) + \|\nabla u(t)\|^2 - \gamma \|u(t)\|^2 - \|u'(t)\|^2 = 0,
\]
(13)
respectively, where we write
\[
E(t) = E_1(t) - \frac{\gamma}{2} \|u(t)\|^2 = \frac{1}{2} \left( \|u'(t)\|^2 + \|\nabla u(t)\|^2 - \gamma \|u(t)\|^2 \right)
\]
for simplicity. Then, taking (12) + \( \varepsilon \times (13) \) for any small \( \varepsilon > 0 \), we have
\[
\frac{d}{dt} F(t) + G(t) = 0
\]
(14)
where \( F(t) \) and \( G(t) \) are defined by
\[
F(t) = E(t) + \varepsilon \left( \omega \|\nabla u(t)\|^2 + \delta \|u(t)\|^2 + 2 \int_{\Omega} u(t)u'(t) \, dx \right)
\]
and
\[
G(t) = (\omega \|\nabla u'(t)\|^2 + \delta \|u'(t)\|^2) + \varepsilon \left( \|\nabla u(t)\|^2 - \gamma \|u(t)\|^2 - \|u'(t)\|^2 \right).
\]
Here, it is easy to see from the Cauchy inequality and the Poincaré inequality that
\[
F(t) \leq C(\|u'(t)\|^2 + \|\nabla u(t)\|^2)
\]
(15)
and
\[
G(t) \geq (\delta + \omega \lambda_1 - \varepsilon)\|u'(t)\|^2 + \varepsilon(1 - \gamma / \lambda_1)\|\nabla u(t)\|^2.
\]
(16)
Thus, if $\delta + \omega \lambda_1 > 0$, choosing small $\varepsilon > 0$, we have from (14)–(16) that
\[
\frac{d}{dt} F(t) + kF(t) \leq 0 \tag{17}
\]
with some constant $k > 0$. Moreover, we observe from the Young inequality and the Poincaré inequality that
\[
F(t) \geq \frac{1}{2} \left( \| u'(t) \|^2 + (1 - \gamma/\lambda_1) \| \nabla u(t) \|^2 \right) \\
+ \frac{\varepsilon}{2} \left( (\delta + \omega \lambda_1) \| u(t) \|^2 - 2 \| u(t) \| \| u'(t) \| \right) \\
\geq \frac{1}{2} (1 - C\varepsilon) \| u'(t) \|^2 + \frac{1}{2} (1 - \gamma/\lambda_1) \| \nabla u(t) \|^2. \tag{18}
\]
Thus, choosing small $\varepsilon > 0$, we have from (17) and (18) that
\[
\| u'(t) \|^2 + \| \nabla u(t) \|^2 \leq CF(t) \leq CF(0)e^{-kt}
\]
for $t \geq 0$. □

References


