Numbers associated to Symmetric Differential Operators and the Bernoulli Numbers

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Abstract

By considering a certain symmetric differential operator we introduce a sequence of numbers $\{C_k\}_{k=0}^{\infty}$, and clarify their properties, which are similar to those of the Bernoulli numbers. It is shown that the generating function of $\{C_k\}$ is the hyperbolic tangent function, and some (maybe known) properties of the Bernoulli numbers are derived through those of C_k.

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Introduction

This is a continuation of the previous note [4], in which we have considered a certain symmetric differential operator and have derived certain properties or identities concerning the binomial coefficients. On the basis of the results in [4] we introduce in this note a sequence of numbers $\{C_k\}_{k=0}^{\infty}$ associated to the coefficients of the operators, and we clarify that these numbers have properties analogous with the Bernoulli numbers.

After reviewing in §1 the results on the symmetric differential operators considered in [4], we introduce in §2 numbers $\{C_k\}$, and investigate their properties. In §3 through the generating function of $\{C_k\}$ we see the relationship between C_k and the Bernoulli numbers, and obtain (maybe rediscover) some properties of the Bernoulli numbers.

1 Symmetric differential operators

Let $C_0^\infty(\mathbb{R})$ denote the space of complex-valued C^∞ functions on $\mathbb{R}$ with compact support. Suppose the space $C_0^\infty(\mathbb{R})$ is endowed with the inner product $(\cdot,\cdot)$ defined by

$$(f,g) := \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx \qquad (f,g \in C_0^\infty(\mathbb{R})).$$

Let D denote the differential operator $\frac{1}{i}\frac{d}{dx}$ $(i := \sqrt{-1})$. Then, D is a *symmetric* operator, namely,

$$(Df,g) = (f,Dg) \qquad (f,g \in C_0^\infty(\mathbb{R}))$$

holds.

We consider the symmetric (or formally self-adjoint) operator whose principal symbol is given by the monomial of degree n given by

$$p_n(x,\xi) = a(x)\xi^n.$$

By applying the corresponding rule:

$$x \mapsto x\,\cdot, \quad \xi \mapsto D,$$

we get the n-th order differential operator

$$Q = a(x)D^n$$

corresponding to $p_n(x,\xi)$. Then, we have the following.

Lemma 1 *The adjoint operator Q^* of Q is given by*

$$Q^* = D^n\big[\,\overline{a(x)}\cdot\,\big] = \sum_{p=0}^{n}\binom{n}{p}\big(D^p\bar{a}(x)\big)D^{n-p}.$$

Thus Q is not a symmetric operator. As a symmetric operator corresponding to $p_n(x,\xi)$ we consider the differential operator

$$P_n = a(x)D^n + \sum_{p=1}^{n} c_p^n\big(D^p a(x)\big)D^{n-p}, \tag{1}$$

where $a(x)$ is a real-valued function, and c_p^n's are complex constants. By virtue of Lemma 1 we have the following.

Lemma 2 *The operator P_n is symmetric, i.e., $P_n^* = P_n$ if and only if the coefficients c_p^n $(p = 1, 2, \dots n)$ satisfy*

$$\begin{aligned} c_p^n &= (-1)^p \bar{c}_p^n + (-1)^{p-1} \binom{n-p+1}{1} \bar{c}_{p-1}^n \\ &\quad + (-1)^{p-2} \binom{n-p+2}{2} \bar{c}_{p-2}^n + \cdots - \binom{n-1}{p-1} \bar{c}_1^n + \binom{n}{p}. \end{aligned} \tag{2}$$

We assume the coefficients c_p^n $(p = 1, 2, \dots, n)$ to be

$$c_p^n = \begin{cases} \text{a real number} & (p : \text{odd}) \\ 0 & (p : \text{even}) \end{cases}. \tag{3}$$

Theorem 3 ([4]) *For any $n \in \mathbb{N}$, and any real valued function $a(x)$ there exists an unique n-th order symmetric differential operator P of the form (1) satisfying the assumption (3).*

Proof. First we show the existence of P (cf. [3, Lemma 4.2]). Let $Q_0 := a(x)D^n$. Put

$$Q_1 := \frac{1}{2}(Q_0 + Q_0^*).$$

Then, by means of Lemma 1 Q_1 is a symmetric operator with the n-th order term being equal to Q_0, and the coefficients

$$\frac{1}{2}\binom{n}{p} D^p a(x)$$

of the $(n-p)$-th order term of Q_1 are real if p is even. Let R_{n-2} denote the $(n-2)$-th order term of Q_1, and put

$$Q_2 := Q_1 - \frac{1}{2}(R_{n-2} + R_{n-2}^*).$$

Then, Q_2 is a symmetric operator of the form (1) with c_p^n being real and $c_2^n = 0$.

Next, let R_{n-4} be the $(n-4)$-th order term of Q_2, and put

$$Q_4 := Q_2 - \frac{1}{2}(R_{n-4} + R_{n-4}^*).$$

Then, Q_4 is a symmetric operator of the form (1) with c_p^n being real and $c_2^n = c_4^n = 0$. Thus by continuing this process we get $Q_2, Q_4, Q_6, \dots$, and we obtain the required operator P_n as Q_{n-1} if n is odd, or Q_n if n is even.

Next, we show that the coefficients c_p^n is uniquely determined by the condition (2) under the assumption (3).

Suppose n is odd. The condition (2) for $p = 1, 2, \dots$ gives a system of linear equations for $c_1^n, c_3^n, \dots, c_{n-2}^n, c_n^n$ as follows:

$$\begin{aligned}
2c_1^n &= \binom{n}{1}, \\
\binom{n-1}{1} c_1^n &= \binom{n}{2}, \\
2c_3^n + \binom{n-1}{2} c_1^n &= \binom{n}{3}, \\
\binom{n-3}{1} c_3^n + \binom{n-1}{3} c_1^n &= \binom{n}{4}, \\
&\cdots\cdots\cdots \\
\binom{2}{1} c_{n-2}^n + \binom{4}{3} c_{n-4}^n + \cdots\cdots + \binom{n-1}{n-2} c_1^n &= \binom{n}{n-1}, \\
2c_n^n + \binom{2}{2} c_{n-2}^n + \binom{4}{4} c_{n-4}^n + \cdots\cdots + \binom{n-1}{n-1} c_1^n &= \binom{n}{n}.
\end{aligned}$$

It is easy to see that the rank of the $(n \times (n+1)/2)$-matrix of the coefficients of the above linear equations is equal to $(n+1)/2$. Hence, the solution (if exists) is unique.

If n is even, the linear equations for $c_1^n, c_3^n, \dots, c_{n-1}^n$ is the following:

$$\begin{aligned}
2c_1^n &= \binom{n}{1}, \\
\binom{n-1}{1} c_1^n &= \binom{n}{2}, \\
2c_3^n + \binom{n-1}{2} c_1^n &= \binom{n}{3}, \\
\binom{n-3}{1} c_3^n + \binom{n-1}{3} c_1^n &= \binom{n}{4}, \\
&\cdots\cdots\cdots \\
2c_{n-1}^n + \binom{3}{2} c_{n-3}^n + \cdots\cdots + \binom{n-1}{n-2} c_1^n &= \binom{n}{n-1}, \\
\binom{1}{1} c_{n-1}^n + \binom{3}{3} c_{n-3}^n + \cdots\cdots + \binom{n-1}{n-1} c_1^n &= \binom{n}{n}.
\end{aligned}$$

This system similarly derives the uniqueness of the solution. ∎

From the above system of linear equations for $c_1^n, c_3^n, \dots$ we have the following.

Theorem 4 ([4]) *Let $1 \leq k \leq (n+1)/2$. The following two systems of linear* *equations for $c_{n-1}, c_{n-3}, \ldots, c_{n-2k+1}$ are equivalent each other :*

$$\begin{bmatrix} 2 & & & & & \\ \binom{n-1}{2} & 2 & & & \mathbf{0} & \\ \binom{n-1}{4} & \binom{n-3}{2} & 2 & & & \\ \vdots & \vdots & & \ddots & & \\ \binom{n-1}{2k-4} & \binom{n-3}{2k-6} & \cdots & \binom{n-2k+5}{2} & 2 & \\ \binom{n-1}{2k-2} & \binom{n-3}{2k-4} & \cdots & \binom{n-2k+5}{4} & \binom{n-2k+3}{2} & 2 \end{bmatrix} \begin{bmatrix} c_1^n \\ c_3^n \\ \vdots \\ \vdots \\ c_{2k-3}^n \\ c_{2k-1}^n \end{bmatrix} = \begin{bmatrix} \binom{n}{1} \\ \binom{n}{3} \\ \vdots \\ \vdots \\ \binom{n}{2k-3} \\ \binom{n}{2k-1} \end{bmatrix}, \quad (4)$$

$$\begin{bmatrix} \binom{n-1}{1} & & & & & \\ \binom{n-1}{3} & \binom{n-3}{1} & & & \mathbf{0} & \\ \binom{n-1}{5} & \binom{n-3}{3} & \binom{n-5}{1} & & & \\ \vdots & \vdots & & \ddots & & \\ \binom{n-1}{2k-3} & \binom{n-3}{2k-5} & \cdots & \cdots & \binom{n-2k+3}{1} & \\ \binom{n-1}{2k-1} & \binom{n-3}{2k-3} & \cdots & \cdots & \binom{n-2k+3}{3} & \binom{n-2k+1}{1} \end{bmatrix} \begin{bmatrix} c_1^n \\ c_3^n \\ \vdots \\ \vdots \\ c_{2k-3}^n \\ c_{2k-1}^n \end{bmatrix} = \begin{bmatrix} \binom{n}{2} \\ \binom{n}{4} \\ \vdots \\ \vdots \\ \binom{n}{2k-2} \\ \binom{n}{2k} \end{bmatrix}. \quad (5)$$

Applying Cramer's formulas for the solution c_{2k-1}^n of (4) and (5), we have the following.

Corollary 5 *For $n, k \in \mathbb{N}$ with $1 \le k \le (n+1)/2$ we have*

$$c_{2k-1}^n = \frac{(-1)^{k-1}}{2^k} \begin{vmatrix} \binom{n}{1} & 2 & & & \\ \binom{n}{3} & \binom{n-1}{2} & 2 & & \mathbf{0} \\ \binom{n}{5} & \binom{n-1}{4} & \binom{n-3}{2} & \ddots & \\ \vdots & \vdots & \vdots & & 2 \\ \binom{n}{2k-1} & \binom{n-1}{2k-2} & \binom{n-3}{2k-4} & \cdots & \binom{n-2k+3}{2} \end{vmatrix} \quad (6)$$

$$= (-1)^{k-1} \frac{(n-2k-1)!!}{(n-1)!!} \begin{vmatrix} \binom{n}{2} & \binom{n-1}{1} & & & \\ \binom{n}{4} & \binom{n-1}{3} & \binom{n-3}{1} & & \mathbf{0} \\ \binom{n}{6} & \binom{n-1}{5} & \binom{n-3}{3} & \ddots & \\ \vdots & \vdots & \vdots & & \binom{n-2k+3}{1} \\ \binom{n}{2k} & \binom{n-1}{2k-1} & \binom{n-3}{2k-3} & \cdots & \binom{n-2k+3}{3} \end{vmatrix}. \quad (7)$$

Here the formula means $c_1^n = \frac{1}{2}\binom{n}{1} = \frac{1}{n-1}\binom{n}{2}$ *if* $k=1$.

2 Sequence of numbers associated to c_p^n

We can calculate c_p^n by the formula (6) or (7) and get Table 1 for small n and p. By observing Table 1 we present and can prove the following proposition.

Proposition 6 *We have a sequence of numbers* $\{C_k\}_{k=1}^{\infty}$ *which satisfies*

$$c_k^n = \binom{n}{k} C_k, \tag{8}$$

for $n, k \in \mathbb{N}$ *with* $1 \le k \le n$.

Table 1: c_p^n

n	c_1^n	c_2^n	c_3^n	c_4^n	c_5^n	c_6^n	c_7^n	c_8^n	c_9^n	c_{10}^n
1	$\frac{1}{2}$									
2	1	0								
3	$\frac{3}{2}$	0	$-\frac{1}{4}$							
4	2	0	-1	0						
5	$\frac{5}{2}$	0	$-\frac{5}{2}$	0	$\frac{1}{2}$					
6	3	0	-5	0	3	0				
7	$\frac{7}{2}$	0	$-\frac{35}{4}$	0	$\frac{21}{2}$	0	$-\frac{17}{8}$			
8	4	0	-14	0	28	0	-17	0		
9	$\frac{9}{2}$	0	-21	0	63	0	$-\frac{153}{2}$	0	$\frac{31}{2}$	
10	5	0	-30	0	126	0	-255	0	155	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Proof. We have $C_{2m} = 0$ $(m \in \mathbb{N})$ because $c_{2m}^n = 0$. We show (8) for $k = 2m-1$ by induction with respect to m. (i) $c_1^n = \binom{n}{1}(1/2)$, i.e., $C_1 = 1/2$. (ii) Suppose

$$c_{2j-1}^n = \binom{n}{2j-1} C_{2j-1}$$

for $0 \le j \le m-1$. It follows from the last equation of the system (4) that

$$c_{2m-1}^n = -\frac{1}{2}\sum_{j=1}^{m-1}\binom{n-2j+1}{2m-2j} c_{2j-1}^n + \frac{1}{2}\binom{n}{2m-1}.$$

Hence

$$c^n_{2m-1} = -\frac{1}{2}\sum_{j=1}^{m-1}\binom{n-2j+1}{2m-2j}\binom{n}{2j-1}C_{2j-1} + \frac{1}{2}\binom{n}{2m-1}.$$

Here note that

$$\begin{aligned}\binom{n-2j+1}{2m-2j}\binom{n}{2j-1} &= \frac{(n-2j+1)!}{(2m-2j)!(n-2m+1)!}\frac{n!}{(2j-1)!(n-2j+1)!}\\ &= \frac{n!}{(n-2m+1)!(2m-1)!}\frac{(2m-1)!}{(2m-2j)!(2j-1)!}\\ &= \binom{n}{2m-1}\binom{2m-1}{2j-1},\end{aligned}$$

and we have

$$c^n_{2m-1} = \binom{n}{2m-1}\left\{\frac{1}{2} - \frac{1}{2}\sum_{j=1}^{m-1}\binom{2m-1}{2j-1}C_{2j-1}\right\} = \binom{n}{2m-1}C_{2m-1},$$

where

$$C_{2m-1} = \frac{1}{2} - \frac{1}{2}\sum_{j=1}^{m-1}\binom{2m-1}{2j-1}C_{2j-1}. \tag{9}$$

■

By the formula (8) we have $C_p = c^p_p$, and accordingly see $C_1, C_2, C_3, \dots$ to be diagonal elements of Table 1. We also find from (6) that

$$C_{2k-1}\left(= c^{(2k-1)}_{(2k-1)}\right) = \frac{(-1)^{k-1}}{2^k}\begin{vmatrix} \binom{2k-1}{1} & 2 & & & \\ \binom{2k-1}{3} & \binom{2k-2}{2} & 2 & & \text{\Large 0} \\ \binom{2k-1}{5} & \binom{2k-2}{4} & \binom{2k-4}{2} & \ddots & \\ \vdots & \vdots & \vdots & & 2 \\ \binom{2k-1}{2k-1} & \binom{2k-2}{2k-2} & \binom{2k-4}{2k-4} & \cdots & \binom{2}{2} \end{vmatrix} \tag{10}$$

for $k = 1, 2, 3, \dots$ (Table 2).

As a result we have a representation of the symmetric differential operator P_n by means of the numbers $\{C_p\}$:

$$P_n = a(x)D^n + \sum_{p=1}^{n}\binom{n}{p}C_p(D^p a(x))D^{n-p}.$$

We put $C_0 = -1$. Then, we have the following theorem concerning the recurrence relation for C_p.

Table 2: C_p

p	1	3	5	7	9	11	13	15	17
C_p	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{17}{8}$	$\frac{31}{2}$	$-\frac{691}{4}$	$\frac{5461}{2}$	$-\frac{929569}{16}$	$\frac{3202291}{2}$

Theorem 7 (Recurrence relation) *The sequence of numbers* $\{C_k\}_{k=0}^{\infty}$ *is given by the following recurrence relation:*

$$\sum_{j=0}^{k-1} \binom{k}{j} C_j + 2C_k = 0 \quad (k \geq 1), \qquad C_0 = -1, \tag{11}$$

or equivalently,

$$C_k = -\sum_{j=0}^{k} \binom{k}{j} C_j \ \ (k \geq 1), \quad C_0 = -1. \tag{12}$$

Proof. Put $n = 2k$ in the last equation of the system (5), and we obtain

$$\sum_{j=1}^{k} c_{2j-1}^{2k} = 1.$$

This derives the relation:

$$\binom{2k}{0} C_0 + \sum_{j=1}^{k} \binom{2k}{2j-1} C_{2j-1} = 0 \quad (k \geq 1). \tag{13}$$

On the other hand, from (9) we have

$$\binom{2k-1}{0} C_0 + \sum_{j=1}^{k-1} \binom{2k-1}{2j-1} C_{2j-1} + 2C_{2k-1} = 0 \quad (k \geq 1). \tag{14}$$

We see that the relations (13) and (14) with

$$C_{2k} = 0 \ (k \geq 1)$$

are equivalent to the relation (11). ■

Next we consider the denominator of C_p, and obtain the following.

Theorem 8 *(1) For an integer $k \geq 1$ put $2k = 2^{\alpha} q$ with q being an odd integer. Then, $2^{\alpha} C_{2k-1}$ is an odd integer, i.e., the denominator of C_{2k-1} is equal to 2^{α}.*

(2) The coefficients

$$c_{2k-1}^{2m} = \binom{2m}{2k-1} C_{2k-1} \quad (1 \le k \le m)$$

of the differential operator P_{2m} are integers.

Proof. We see by the formula (10) that the denominator of C_k is 2^α for some non-negative integer α. If

$$2k = \binom{2k}{2k-1} = \frac{2k(2k-1)(2k-2)\cdots 2}{(2k-1)!} = 2^\alpha q,$$

$$\binom{2m}{2k-1} = \frac{2m(2m-1)\cdots(2m-2k+2)}{(2k-1)!} = 2^\beta q' \quad (m > k),$$

where q and q' are odd integers, then we have

$$\begin{aligned} \binom{2m}{2k-1} \Big/ \binom{2k}{2k-1} &= 2^{\beta-\alpha} \cdot \frac{q'}{q} \\ &= \frac{2m(2m-1)\cdots(2m-2k+2)}{2k(2k-1)(2k-2)\cdots 2} \\ &= \frac{2^k \cdot m(m-1)\cdots(m-k+1)\cdot q_1}{2^k \cdot k(k-1)\cdots 1 \cdot q_2} = \binom{m}{k} \cdot \frac{q_1}{q_2}, \end{aligned} \tag{15}$$

where q_1, q_2 are odd integers. Since $\binom{m}{k}$ is an integer, $\beta \ge \alpha$ holds. Hence, the fact that $\binom{2m}{2k-1} C_{2k-1}$ to be an integer follows from the fact that $2^\alpha C_{2k-1}$ is an (odd) integer, namely the assertion (2) follows from the assertion (1).

We obtain from (15) that

$$\binom{2m}{2k-1} C_{2k-1} = \binom{m}{k} \cdot \frac{q_1}{q_2} \cdot \binom{2k}{2k-1} C_{2k-1} = \binom{m}{k} \cdot \frac{q_1}{q_2} \cdot 2^\alpha q C_{2k-1},$$

and accordingly find that

$$\binom{2m}{2k-1} C_{2k-1} \text{ is odd (resp. even)} \iff \binom{m}{k} \text{ is odd (resp. even)} \tag{16}$$

for $1 \le k < m$ if $2^\alpha C_{2k-1}$ is an odd integer.

We will show by induction with respect to integers k that $2^\alpha C_{2k-1} = \binom{2k}{2k-1} C_{2k-1}$ is odd. (i) For $k = 1$ the assertion holds as $2C_1 = 1$. (ii) Suppose $2^\beta C_{2l-1}$ $(2l = 2^\beta \times \text{(an odd integer)})$ is odd for $1 \le l < k$. Notice the formula

$$\binom{2k}{2k-1} C_{2k-1} = 1 - \binom{2k}{1} C_1 - \binom{2k}{3} C_3 - \cdots - \binom{2k}{2k-3} C_{2k-3},$$

and we have to show that

$$\binom{2k}{1}C_1 + \binom{2k}{3}C_3 + \cdots + \binom{2k}{2k-3}C_{2k-3}$$

is even. This is shown by virtue of (16) and the fact that $\sum_{l=1}^{k-1}\binom{k}{l}$ is even $(= 2^k - 2)$. ■

Review on Bernoulli numbers (see [1], [2, §6.5], for example).

Let us consider the sum of kth powers

$$S_k(n) = 1^k + 2^k + \cdots + n^k.$$

By summing the formulas:

$$(m+1)^{k+1} - m^{k+1} = \sum_{j=0}^{k}\binom{k+1}{j}m^j$$

for $m = 1, 2, \ldots, n$, we get

$$(n+1)^{k+1} - 1 = \sum_{j=0}^{k}\binom{k+1}{j}S_j(n),$$

namely,

$$S_k(n) = \frac{1}{k+1}\Big\{(n+1)^{k+1} - 1 - \sum_{j=0}^{k-1}\binom{k+1}{j}S_j(n)\Big\}. \tag{17}$$

Noticing $S_0(n) = n$ we see by induction that $S_k(n)$ is given as

$$S_k(n) = \sum_{j=0}^{k} s_j^k n^{k+1-j} \qquad \text{with} \quad s_0^k = \frac{1}{k+1}.$$

For the coefficients s_n^k there exists a sequence of numbers $\{B_j\}_{j=0}^{\infty}$ such that

$$s_j^k = \frac{(-1)^j}{k+1}\binom{k+1}{j}B_j \quad (0 \le j \le k).$$

The numbers B_j are called *Bernoulli numbers*. Hence, we have

$$S_k(n) = \frac{1}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k+1}{j}B_j n^{k+1-j}. \tag{18}$$

By virtue of (17) we find that $\{B_k\}_{k=0}^{\infty}$ satisfy the recurrence relation

$$B_k = \sum_{j=0}^{k} \binom{k}{j} B_j \ (k \geq 2) \quad \text{with} \quad B_0 = 1,\ B_1 = -1/2. \tag{19}$$

Comparing (12) and (19) we remark that $\{C_k\}$ and $\{B_k\}$ are closely related.

3 Generating function - Relationship with Bernoulli numbers

We consider the generating function for the sequence $\{C_k\}$, which gives the explicit relationship with the Bernoulli numbers.

Proposition 9 (Exponential generating function) *We have*

$$\sum_{k=0}^{\infty} \frac{C_k}{k!} z^k = -\frac{2}{e^z+1} = \tanh\left(\frac{z}{2}\right) - 1 \qquad (|z| < \pi). \tag{20}$$

Proof. Put

$$F(z) := \sum_{k=0}^{\infty} \frac{C_k}{k!} z^k.$$

Then, we have formally

$$\begin{aligned}
e^z F(z) &= \Big(\sum_{j=0}^{\infty} \frac{1}{j!} z^j\Big)\Big(\sum_{k=0}^{\infty} \frac{C_k}{k!} z^k\Big) \\
&= C_0 + (C_0 + C_1)z + \cdots \\
&\quad + \left(\frac{C_0}{0!(2k-1)!} + \frac{C_1}{1!(2k-2)!} + \frac{C_3}{3!(2k-2)!} + \cdots + \frac{C_{2k-1}}{(2k-1)!0!}\right) z^{2k-1} \\
&\quad + \left(\frac{C_0}{0!(2k)!} + \frac{C_1}{1!(2k-1)!} + \frac{C_3}{3!(2k-3)!} + \cdots + \frac{C_{2k-1}}{(2k-1)!1!}\right) z^{2k} \\
&\quad + \cdots \\
&= -1 - \frac{1}{2} z + \cdots \\
&\quad + \frac{1}{(2k-1)!} \left\{ \binom{2k-1}{0} C_0 + \binom{2k-1}{1} C_1 + \cdots + \binom{2k-1}{2k-1} C_{2k-1} \right\} z^{2k-1} \\
&\quad + \frac{1}{(2k)!} \left\{ \binom{2k}{0} C_0 + \binom{2k}{1} C_1 + \cdots + \binom{2k}{2k-1} C_{2k-1} \right\} z^{2k} \\
&\quad + \cdots \\
&= -2 - F(z).
\end{aligned}$$

Here the last equality follows form the formulas (13) and (14). As a consequence we get the assertion. ■

Corollary 10 *For any $n \geq 2$, we have*

$$\sum_{j=1}^{n} \binom{n}{j} C_j B_{n-j} = 0. \tag{21}$$

Particularly, we have

$$\sum_{j=0}^{m} \binom{2m+1}{2j+1} C_{2j+1} B_{2m-2j} = 0 \qquad (m \geq 1). \tag{22}$$

Proof. We have only to show (21) for odd n, that is the formula (22), because $C_j B_{n-j} = 0$ for any $j \geq 1$ if n is even. Note that

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k,$$

and

$$\Big(-\frac{2}{e^z + 1} \Big) \Big(\frac{z}{e^z - 1} \Big) = -\frac{2z}{e^{2z} - 1}.$$

Hence, we have

$$\left(\sum_{j=0}^{\infty} \frac{C_j}{j!} z^j \right) \left(\sum_{k=0}^{\infty} \frac{B_k}{k!} z^k \right) = -\sum_{n=0}^{\infty} \frac{B_n}{n!} (2z)^n.$$

If $n (\geq 2)$ is odd, i,e, $n = 2m + 1$, then $B_n = 0$, hence we have

$$\sum_{j=0}^{n} \frac{C_j B_{n-j}}{j!(n-j)!} = 0,$$

which leads the formula (21). ■

Corollary 11 *We have*

$$C_k = \frac{2}{k+1} (2^{k+1} - 1) B_{k+1} \quad (k \geq 0). \tag{23}$$

Proof. We have

$$\begin{aligned} -\frac{2}{e^z + 1} &= -2 \left(\frac{1}{e^z - 1} - \frac{2}{e^{2z} - 1} \right) \\ &= -\frac{2}{z} \left\{ \Big(1 + \sum_{k=1}^{\infty} \frac{B_k}{k!} z^k \Big) - \Big(1 + \sum_{k=1}^{\infty} \frac{B_k}{k!} (2z)^k \Big) \right\} \\ &= -1 + \sum_{k=2}^{\infty} 2(2^k - 1) B_k \frac{z^{k-1}}{k!}. \end{aligned}$$

Therefore we obtain the formula (23). ■

From (21) and (23) we can derive the following relation between Bernoulli numbers by using the identity $\frac{1}{j+1}\binom{n}{j} = \frac{1}{n+1}\binom{n+1}{j+1}$.

Proposition 12 *For $n \geq 4$ we have*

$$\sum_{j=2}^{n} \binom{n}{j} (2^j - 1) B_j B_{n-j} = 0. \tag{24}$$

By combining this theorem with the formula (23) we obtain the following property concerning the Bernoulli numbers.

Proposition 13 *Let $n (\geq 2)$ be an even integer, and given by $n = 2^\alpha q$ with q being an odd integer. Then,*

$$\frac{2(2^n - 1)}{q} B_n \tag{25}$$

is an odd integer. Moreover,

$$\binom{2m}{n-1} \frac{2(2^n - 1)}{n} B_n \tag{26}$$

is an integer for any $m \geq n/2\ (\geq 1)$.

Remark. The first part of this proposition has been shown by Worpitzky [5, p.232].

4 Concluding Remark

Similarly to the Bernoulli polynomial we define a polynomial

$$C_n(x) = \sum_{k=0}^{n} \binom{n}{k} C_k x^{n-k} = -x^n + \sum_{j=1}^{n} c_j^n x^{n-j}.$$

Then, we have $C_k = C_k(0)$ and see that

$$\sum_{n=0}^{\infty} C_n(x) \frac{z^n}{n!} = -\frac{2e^{xz}}{e^z + 1}$$

from Proposition 9. On the other hand, the polynomials $E_n(x)$ defined by

$$\frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}$$

are called *Euler polynomials* (cf. [2, pp.573-574]). Thus we have

$$C_n(x) = -E_n(x), \qquad C_k = -E_k := -E_k(0).$$

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