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Signed Graphs and Hushimi Trees

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Abstract

The operation of local switching is introduced by Cameron, Seidel and Tsaranov. It acts on the set of all signed graphs on n vertices. In this paper, mainly, we study how local switching acts on trees. We show that two trees on the same vertices are isomorphic if and only if one is transformed to the other by a sequence of local switchings. There is a correspondence between signed graphs and a root lattice. Any signed graph corresponding to the lattice A_n is transformed by a sequence of local switchings to the tree which is regarded as the Dynkin diagram of A_n.

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Introduction

Following [?], we state basic facts about signed graphs. A graph $G = (V, E)$ consists of an n-set V(the vertices) and a set E of unordered pairs from V(the edges). A *signed graph* (G, f) is a graph G with a signing $f : E \to \{1, -1\}$ of the edges. We set $E^+ = f^{-1}(+1)$ and $E^- = f^{-1}(-1)$. For any subset $U \subseteq V$ of vertices, let f_U denote the signing obtained from f by reversing the sign of each edge which has one vertex in U. This defines on the set of signings an equivalence relation, called *switching*. The equivalence classes $\{f_U : U \subseteq V\}$ are the *signed switching classes* of the graph $G = (V, E)$. The *adjacency matrix* $A = (A_{ij})$ is defined by $A_{ij} = f(\{i, j\})$ for $\{i, j\} \in E$; else $A_{ij} = 0$ otherwise. The matrix $2I + A$ is called the *intersection matrix* , and interpreted as the Gram matrix of the inner product of n base vectors $a_1, \cdots, a_n$ in a (possibly indefinite) n-dimensional inner product space $R^{p,q}$. These vectors are roots(which have length $\sqrt{2}$) at angles $\pi/2, \pi/3$, or $2\pi/3$. Their integral linear combinations form a root lattice (an even integral lattice spanned by vectors of norm 2), which we denote by $L(G, f)$. The reflection w_i in the hyperplane orthogonal to the root a_i is given by

$$w_i(x) = x - \frac{2(a_i, x)}{(a_i, a_i)} a_i = x - (x, a_i)a_i.$$

The Weyl group $W(\Gamma, f)$ of $L(G, f)$ is generated by the reflections w_i, $(i = 1, \cdots n)$.

Let $i \in V$ be a vertex of G, and $V(i)$ be the set of neighbours of i. The *local graph* of (G, f) at i has $V(i)$ as its vertex set, and as edges all edges $\{j, k\}$ of G for which $f(i, j)f(j, k)f(k, i) = -1$. A *rim* of (G, f) at i is any union of connected components of local graph at i. Let J be any rim at i, and let $K = V(i)\backslash J$. *Local switching* of (G, f) with respect to (i, J) is the following operation:(i) delete all edges of G between J and K; (ii) for any $j \in J$, $k \in K$ not previously joined, introduce an edge $\{j, k\}$ with sign chosen so that $f(i, j)f(j, k)f(k, i) = -1$; (iii) change the signs of all edges from i to J; (iv) leave all other edges and signs unaltered. Let Ω_n be the set of switching classes of signed graphs of order n. Local switching , applied to any vertex and any rim at the vertex, gives a relation on Ω_n which is symmetric but not transitive. The equivalence classes of its transitive closure are called the *clusters* of order n. If two signed graphs (G_1, f_1) and (G_2, f_2) are in the same cluster, we say that (G_1, f_1) and (G_2, f_2) are *equivalent by local switching.* They are equivalent by local switching if and only if (G_1, f_1) is transformed to (G_2, f_2) by a sequence of switchings and local switchings.

Let L be a root lattice , B the set of ordered root bases, and B^* the subset of B consisting of bases which arise from signed graphs. Then, $(a_1, \cdots, a_n) \in \mathsf{B}^*$ if and only if $(a_i, a_j) \in \{0, +1, -1\}$ for all $i \neq j$. Many natural operations on B do not preserve B^*. Consider the map

$$\sigma_{ij} : (a_1, \cdots, a_n) \mapsto (a_1, \cdots, w_i(a_j), \cdots, a_n).$$

For any i, the map σ_{ii} just changes the sign of the vector a_i. Hence, they generate the equivalence relation induced by switching and preserve B^*. If i and j are non-adjacent, then σ_{ij} is the identity. So assume that i and j are adjacent. By switching, we may ensure that $(a_i, e_k) \geq 0$ for all k. Then $(a_i, a_j) = 1$ and $(w_i(a_j), a_k) = (a_j, a_k) - (a_i, a_k)$. Hence, if $(a_i, a_k) = 1$ and $(a_j, a_k) = -1$, B^* is not preserved by σ_{ij}. However the product of the commuting maps σ_{ij} and σ_{ik} preserve B^*. Let J be any set of neighbours of i and let $(a_1, \cdots, a_n)$ be a root base in B^*. Then $\prod_{j \in J} \sigma_{ij}$ maps $(a_1, \cdots, a_n)$ to a base in B^* if and only if J is a rim at i. This is the reason why the notion of local switching is defined as above.

We investigate how local switching acts on trees. For this purpose, we need to treat with Hushimi trees. In section 2, we discuss Hushimi trees which are related to the lattice A_n. We show in section 3 that these Hushimi trees are equivalent by local switching to trees with only two leaves. In section 4, we prove that two trees are equivalent by local switching if and only if one is obtained by rearrangement of vertices of the other. We deal with signed cycles in section 5. A signed cycle with odd parity is equivalent to a tree which may be regarded as the Dynkin diagram $[D_n]$ of the lattice D_n. Any signed graph corresponding to the lattice D_n is also equivalent to the tree $[D_n]$.

1. The lattice A_n and signed Hushimi trees

A connected graph $G = (V, E)$ is called *Hushimi tree* if each block of G is a complete graph. A complete graph is a Hushimi tree of one block. Let a be a cut-vertex of a Hushimi tree G. If G is divided into m connected components when the cut-vertex a is deleted, in the present paper, we say that *the Hushimi degree(simply H-degree) of the cut-vertex a is m* . If a vertex a of G is not a cut-vertex, its H-degree is defined to be 1. A connected subgraph of a Hushimi tree G is called a *sub-Hushimi tree* if it consists of some blocks of G. A block of Hushimi tree is said to be *pendant* if it has only one cut-vertex. It is evident that any Hushimi tree has at least two pendant blocks.

Definition. In this paper, a Hushimi tree is said to be *simple* if the H-degree of any its cut-vertex is 2. A Hushimi tree is said to be *semi-simple* if its each block has at most two cut-vertices whose H-degree are greater than 2. A signed Hushimi tree is called a Hushimi tree *with positive sign* (or simply a *positive* Hushimi tree) if we can switch all signs of edges into $+1$. A tree with only two leaves is said to be *a line-tree.*

A tree is always considered as a Hushimi tree with positive sign. The lattice A_n is spanned by vectors $e_i - e_j$, $1 \leq i \neq j \leq n+1$, where $\{e_1, \cdots, e_{n+1}\}$ is the orthonormal base of the euclidean $(n+1)$-space R^{n+1}. There is the one-to -one correspondence between ordered root bases of A_n and connected signed graphs associated with A_n. A line-tree with n vertices may be considered as the Dynkin diagram $[A_n]$ of the lattice A_n.

Theorem 1. Any connected signed graph is a signed graph associated with A_n if and only if it is a positive simple Hushimi tree.

Proof. Let G be a signed graph corresponding to an ordered base $\{a_1, a_2, \cdots, a_n\}$ of the lattice A_n. If we replace a_i by $-a_i$, then the sign of G is switched with respect to $\{a_i\}$. Hence there is no problem whether we take a_i or $-a_i$. There are no induced cycles in G whose length are more than 3. In fact, if $a_{i_1}, a_{i_2}, \cdots, a_{i_m}, (m > 3)$ make an induced cycle, then we can assume that $a_{i_1} = e_{j_1} - e_{j_2}, a_{i_2} = e_{j_2} - e_{j_3}, a_{i_3} = e_{j_3} - e_{j_4}, \cdots, a_{i_m} = e_{j_m} - e_{j_1}$. But this implies that $a_{i_1}, a_{i_2}, \cdots, a_{i_m}$ are not linearly independent. If $a_{i_1}, a_{i_2}, a_{i_3}$ make an induced cycle, then we can assume that $a_{i_1} = e_j - e_{j_1}, a_{i_2} = e_j - e_{j_2}$, $a_{i_3} = e_j - e_{j_3}$. We have induced cycles of this type only in G. Now take a block B of G consisting of vertices $a_{i_1}, a_{i_2}, \cdots, a_{i_m}$. Two vertices a_{i_1} and a_{i_2} must be on an induced cycle in B. We may assume that $a_{i_1}, a_{i_2}, a_{i_3}$ make an induced cycle. Then we can put $a_{i_1} = e_j - e_{j_1}, a_{i_2} = e_j - e_{j_2}, a_{i_3} = e_j - e_{j_3}$. Two vertices a_{i_1} and a_{i_4} are also on an induced cycle in B, which may consist of $a_{i_1}, a_{i_4}, a_{i_5}$. Then we can put $a_{i_4} = e_j - e_{j_4}, a_{i_5} = e_j - e_{j_5}$ or $a_{i_4} = e_{j_1} - e_{j_4}, a_{i_5} = e_{j_1} - e_{j_5}$, where $j_4 \neq j$. Assume that $a_{i_4} = e_{j_1} - e_{j_4}, a_{i_5} = e_{j_1} - e_{j_5}$. Two vertices a_{i_2} and a_{i_4} are also on an induced cycle in B. Then we have $j_4 = j_2$, a contradiction. Hence, we get $a_{i_4} = e_j - e_{j_4}, a_{i_5} = e_j - e_{j_5}$. By this way, we get $a_{i_k} = e_j - e_{j_k}$, $1 \leq k \leq m$. Hence any block of G is a complete graph whose edges have sign $+ 1$. Suppose that the vertex $a_{i_1} = e_j - e_{j_1}$ of a block B is a cut-vertex. If two vertex a_k, a_ℓ which are not in B are adjacent with a_{i_1}, then we can put $a_k = e_{j_1} - e_{k_1}$, $a_\ell = e_{j_1} - e_{\ell_1}$. Hence a_{i_1}, a_k, a_ℓ are contained in another block of G. Hence we show that $G - a_{i_1}$ has two connected components. Thus G is a Hushimi tree with positive sign and the H-degree of any cut-vertex of G is 2.

Conversely, let G be a positive Hushimi tree whose any cut-vertex has the H-degree 2. Assume that G has m blocks. If $m = 1$, it is evident that G is a connected signed graph associated with A_n. Now suppose that the result is true for positive Hushimi trees with m blocks whose any cut-vertex has the H-degree 2. Let G be a positive Hushimi tree with $m + 1$ blocks. Let B be a pendant block of G and $a_1 = e_{i_1} - e_{i_2}$ be its cut-vertex. Let G' be the positive Hushimi tree which is made from G by deleting $B \setminus \{a_1\}$. Then G' is a connected signed graph associated with A_n and corresponding to an ordered base $\{a_1, a_2, \cdots, a_n\}$, where A_n is spanned by vectors $e_i - e_j$, $1 \leq i \neq j \leq n + 1$. Now, all the ℓ vertices of B are adjacent with a vertex $a_1 = e_{i_1} - e_{i_2}$. We can assume that e_{i_2} is not used in any other a_j. Then, we can consider that the block B consists of $e_{i_2} - e_{n+2}, e_{i_2} - e_{n+3}, \cdots, e_{i_2} - e_{n+\ell+1}$ and a_1, where $\{e_1, \cdots, e_{n+1}, e_{n+2}, \cdots, e_{n+\ell+1}\}$ is the orthonormal base of the euclidean $(n + \ell + 1)$-space $R^{n+\ell+1}$. Hence we regard G as a connected signed graph associated with $A_{n+\ell}$.

3. Line-trees

Theorem 2 A complete graph with positive sign is equivalent to a line-tree by local switching.

We will prove a little stronger result as follows.

Lemma 3. Let G be a complete graph with positive sign. Take any two vertices a and b of G. Then it can be transformed to a line-tree, by a sequence of local switchings, without adopting local switchings at a and b. Conversely, any line-tree is transformed to a complete graph with positive sign, by a sequence of local switchings, without adopting local switchings at its two leaves.

Proof. Let G consist of vertices $a_1, a_2, \cdots, a_k$. We may assume that $a = a_1$ and $b = a_n$. Set $J = \{a_1\}$ and $K = \{a_3, a_4, \cdots, a_k\}$. By local switching with respect to (a_2, J), we obtain a positive Hushimi tree with two blocks $\{a_1, a_2\}$ and $\{a_2, \cdots, a_k\}$. Next, set $J = \{a_2\}$ and $K = \{a_4, \cdots, a_k\}$. By local switching with respect to (a_3, J), we obtain a positive Hushimi tree with three blocks $\{a_1, a_2\}$, $\{a_2, a_3\}$ and $\{a_3, \cdots, a_k\}$. By this way, we can get a line-tree , by a sequence of local switchings, without adopting local switchings at a_1 and a_n. The converse is obtained by the reverse sequence of local switchings.

We show

Theorem 4. Let G be a positive simple Hushimi tree. Then G is equivalent to a line-tree by local switching. Conversely, a line-tree is transformed to a positive simple Hushimi tree, by any sequence of local switchings.

Firstly, we prepare two lemmas for proving the above theorem.

Lemma 5. *Let G be a positive Hushimi tree consisting of two blocks, B_1 and B_2. Then, it can be transformed to a positive complete graph, by local switching.*

Proof. We can set $B_1 = \{a_1, a_2, \cdots, a_m\}$ and $B_2 = \{a_1, b_1, \cdots, b_k\}$. Then the vertex a_1 is the cut-vertex. Put $J = \{a_2, a_3, \cdots, a_m\}$ and $K = \{b_1, b_2, \cdots, b_k\}$. By local switching with respect to (a_1, J), G is transformed to a complete graph.

Lemma 6. Let G be a positive Hushimi tree. If a is a vertex of G with H-degree 1(resp. 2), then, by local switching, from G, we get a positive Hushimi tree, in which the H-degree of a is 2(resp. 1) and the H-degrees of all the other vertices are not altered.

Proof. Take any vertex a of G. If the H-degree of the vertex a is 2, then there are two blocks B_1 and B_2 which contain a. By local switching at the vertex a, we join B_1 and B_2 and get a positive Hushimi tree where the H-degree of the vertex a is 1 and the H-degrees of all the other vertices are not altered. If the H-degree of a is 1, then there is a block B which contains a. By local switching at a, B is divided into two blocks B_1 and B_2 which contain a. The vertex a has H-degree 2 as a vertex of the new positive Hushimi tree. The H-degrees of all the other vertices are not altered in this case either.

Proof of Theorem 4. If G has only one block, we get the result by Theorem 2. Suppose the result is true for any positive Hushimi tree with m blocks which satisfies the assumption. Now, assume that G has $m+1$ blocks. Take a pendant block B_1 of G with cut-vertex b. Let B_2 be the other block with cut-vertex b. Put $i = b, J = B_1 \backslash b, K = B_2 \backslash b$. By local switching with respect to (b, J), we obtain a positive Hushimi tree with m blocks, which can be transformed to a positive complete graph, by a sequence of local switchings.

It follows from lemma 6 that a positive Hushimi tree whose any cut-vertex has H-degree 2 is transformed to a positive Hushimi tree whose any cut-vertex has H-degree 2, by any local switching. As a line-tree is a positive Hushimai tree whose any cut-vertex has H-degree 2, we get easily that a line-tree is transformed to a positive Hushimi tree whose any cut-vertex has H-degree 2, by any sequence of local switchings.

3. Trees

We show the following results in this section.

Theorem 7. Let G be a positive semi-simple Hushimi tree. Then, G is equivalent to a tree by local switching. Conversely, if a tree is transformed to a positive Hushimi tree G by a sequence of local switchings, then, G is a positive semi-simple Hushimi tree.

Let T be a tree with vertices $\{a_1, \cdots, a_n\}$. Let $\alpha = (a_{i_1}, \cdots, a_{i_n})$ be a permutation of $\{a_1, \cdots, a_n\}$. For each $j, 1 \leq j \leq n$, by replacing a_j with a_{i_j} we get a new tree T' from T. We call T' a *permutation* of T. It is evident that T' is isomorphic to T.

Theorem 8. A tree T_1 is equivalent to a tree T_2 by local switching if and only if T_2 is a permutation of T_1.

From lemma 6, the following is evident.

Lemma 9. Let G be a positive Hushimi tree. Then, it can be transformed to a positive Hushimi tree which has no cut-vertex with H-degree 2, by a sequence of local switchings.

Lemma 10. Let G be a positive Hushimi tree which consists of k-blocks $B_1, \cdots, B_k$,$(k \geq 3)$ and has a unique cut-vertex v contained in all blocks. Hence, the H-degree of v is k. Take any vertex a in one block and b in another block which are not the cut-vertex v. By any sequence of local switchings, we can not construct a complete block containing both a and b. Hence, any positive Hushimi tree which is equivalent to G by local switching and has no cut-vertices with H-degree 2 is isomorphic to G.

Proof. Firstly, let $k = 3$. We may assume $a \in B_1\backslash\{v\}$ and $b \in B_2\backslash\{v\}$. Take any vertex $c \in B_3\backslash\{v\}$.

Case 1. To construct a block containing a and b, from G, we get a signed graph G_1 by local switching with respect to $(\mathrm{v}, J = B_1\backslash\{v\})$. Then, there are the edges ac, ab, but there is no edge bc. To get a complete block containing a and b, we need to join b to c or delete the edge ac (or ab) by local switching. Firstly, we want to join b to c. Take any vertex $a' \in B_1$. By local switching with respect to $(a', J = B_2 \cup B_1\backslash\{a'\})$, we get a singed graph where there is the edge bc but there is no edge ac if $a' \neq a$ or there is no edge vc if $a' = a$. In fact, in this case, each vertex in $B_2\backslash\{v\}$ is jointed to each vertex in $B_3\backslash\{v\}$ but all the edges between $B_3\backslash\{v\}$ and $B_1\backslash\{a'\}$ are deleted. Hence this signed graph is similar to G_1. Thus, we can not get a complete block containing a and b. Take any vertex $a' \in B_1\backslash\{a\}$. Next, we delete the edge ac by local switching with respect to $(a', J = B_1 \cup B_2\backslash\{a'\})$. Then, we get the edge bc. In the signed graph obtained, any two vertices in $B_1 \cup B_2$ are jointed and each vertex in $B_2\backslash\{v\}$ is jointed to each vertex in $B_3\backslash\{v\}$, but all the edges between $B_1\backslash\{a'\}$ and $B_3\backslash\{v\}$ are deleted. This signed graph is also similar to G_1.

Case 2. Assume that $B_1 = B_{11} \cup B_{12}, B_{11} \cap B_{12} = \emptyset, , a \in B_{11}, b \in B_2, c \in B_3$. By local switching with respect to $(v, J = B_{11}\backslash\{v\}))$, we obtained a signed graph G_2. By the same argument as in Case 1, we can show that there is no complete block containing a, b.

Case 3. Assume that $B_1 = B_{11} \cup B_{12}, B_{11} \cap B_{12} = \emptyset, B_2 = B_{21} \cup B_{22}, B_{21} \cap B_{22} = \emptyset, B_3 = B_{31} \cup B_{32}, B_{31} \cap B_{32} = \emptyset, a \in B_{11}, b \in B_{21}, c \in B_{31}$. By local switching with respect to $(v, J = B_{11} \cup B_{22} \cup B_{32} \backslash \{v\})$, we obtained a signed graph G_3. Then, G_3 has the edges ab, ac, but has no edge bc. By similar discussion about B_{11}, B_{21}, B_{31} as in Case 1, we can not get the three edges ab, ac, bc at the same time. In G_3, each vertex in $B_{21} \backslash \{v\}$ is jointed to each vertex in $G_{32} \backslash \{v\}$ and each vertex in $B_{31} \backslash \{v\}$ is jointed to each vertex in $G_{22} \backslash \{v\}$. Even if we ignore these facts, we can not construct a complete block containing B_{11}, B_{21}, B_{31} by the same reason as in Case 1.

Assume $k = 4$.

Case 4. Let $a \in B_1, b \in B_2, c \in B_3$. Set $B_3' = B_3 \cup B_4$. By local switching with respect to $(v, J = B_1 \backslash \{v\}))$, we get a signed graph G_4. Then, G_4 has the edges ab, ac , but has no edge bc. Even if B_3' was a complete block, we could not construct a complete block containing B_1, B_2, B_3'.

Case 5. Let $a \in B_1, b \in B_2, c \in B_3, d \in B_4$. By local switching with respect to $(v, J = B_1 \cup B_4 \backslash \{v\}))$, we get a signed graph G_5. Then. G_5 has the edges ab, ac, db, dc, but has no edges bc, ad. We show as similarly as in Case 1 that we can not construct a complete block containing a, b by deleting the edge bc. By local switching at some vertex, for example d, we will try to join b and c. By local switching with respect to $(d, J = B_3 \cup B_4 \backslash \{v, d\})$, we get a signed graph. Then, each vertex in $B_2 \backslash \{v\}$ is jointed to each vertex in $B_3 \backslash \{v\}$. But, all the edges jointing v and vertices in $B_3 \backslash \{v\}$ are deleted, and if $B_4 \backslash \{v, d\})$ is not empty, all the edges between $B_2 \backslash \{v\}$ and $B_4 \backslash \{v, d\})$ are deleted. The block containing B_1, B_2, B_3 must contain B_4. But, B_4, B_2, B_3 can not make a complete block as we can show by the same argument for B_1, B_2, B_3 in Case 1.

Case 6. Assume that $B_1 = B_{11} \cup B_{12}, B_{11} \cap B_{12} = \emptyset, B_2 = B_{21} \cup B_{22}$, $B_{21} \cap B_{22} = \emptyset$, $B_3 = B_{31} \cup B_{32}, B_{31} \cap B_{32} = \emptyset, B_4 = B_{41} \cup B_{42}, B_{41} \cap B_{42} = \emptyset, a \in B_{11}, b \in B_{21}, c \in B_{31}, d \in B_{41}$. By local switching with respect to $(v, J = B_{11} \cup B_{41} \cup B_{22} \cup B_{32} \backslash \{v\}))$, we obtained a signed graph G_6. Then. G_6 has the edges ab, ac, db, dc, but has no edges bc, ad. By the same argument as in the case 5, even if we ignore $B_{12}, B_{22}, B_{32}, B_{42}$, we can not construct a complete block containing a, b, c, d.

Assume $k \geq 5$.

Case 7. Let $a \in B_1, b \in B_2, c \in B_3$,. Set $B_3' = B_3 \cup B_4 \cup \cdots \cup B_k$. By local switching with respect to $(v, J = B_1 \backslash \{v\}))$, we get a signed graph G_7. Then, G_7 has the edges ab, ac , but has no edge bc. Even if B_3' was a complete block, we could not construct a complete block containing B_1, B_2, B_3'.

Case 8. Let $a \in B_1, b \in B_2, c \in B_3, d \in B_\ell$, $(\ell \leq k)$. Set $B_4' = B_\ell \cup B_{\ell+1} \cup \cdots \cup B_k$ and $B_3' = B_3 \cup \cdots \cup B_{\ell-1}$. By local switching with respect to $(v, J = B_1 \cup B_4' \backslash \{v\})$, we get a signed graph G_8. Then. G_8 has the edges ab, ac, db, dc, but has no edges bc, ad. Even if B_3' and B_4' were complete blocks, we could not construct a complete block containing a, b, c, d by the same argument in Case 5.

When we apply some local switching, it rather prevents from making a complete block to divide given blocks B_i's . Hence, in any cases, we can not construct a complete block containing vertices a, b.

Proof of Theorem 7. By lemma 9, we may assume that G has no cut-vertices with H-degree 2. Select an arbitrary vertex in each pendant block which is not a cut-vertex. We will show that G can be transformed to a tree, by a sequence of local switchings, without adopting local switchings at the selected vertices. Assume that G has m blocks. If $m = 1$, the result follows from Lemma 3. Now suppose that the result is true for Hushimi trees with m blocks which satisfy the assumption. Let G have $m + 1$ blocks. Take any pendant block B_1 with a cut-vertex b. Let $B_2, \cdots, B_k$ be all the other blocks of G which contain the vertex b. We get k sub-Hushimi trees $G_i (i = 1, \cdots k)$ of G, where each G_i contains B_i. Select b and an arbitrary vertex in each pendant block of G_i which is not a cut-vertex. Then, each G_i can be transformed to a tree, by a sequence of local switchings, without adopting local switchings at the selected vertices. Hence, we show the result for the Hushimi tree G.

Now, take a tree T. Then, it is a positive Hushimi tree and its each block has at most two cut-vertices whose H-degree are greater than 2. By lemma 6, by some sequence of local switchings at vertices with H-degree 2, we obtain from T the positive Hushimi tree G_1 which has no cut-vertices with H-degree 2. Take a cut-vertex v of G_1 whose H-degree is k,$(k \geq 3)$. Let G_2 be a signed graph obtained from G_1 by local switching at v. It is evident that G_2 is not a Hushimi tree. It follows from lemma 10 that by any sequence of local switchings, from G_2, we can not get a positive Hushimi tree which has no cut-vertices with H-degree 2 and is not isomorphic to G_1. Thus, we obtain the desired result.

We need the following lemma to prove theorem 8.

Lemma 11. Assume that a tree T has a vertex {v} with degree k and just k leaves. Let a_1 be one of the leaves and $a_1 a_2 \cdots a_\ell v$ be the path between a_1 and v. Take any vertex a_i, $1 \leq i \leq \ell$. Then, by a sequence of local switchings, from T, we get a new tree T' where v and a_i are interchanged and all the other vertices are not altered.

Proof. By a sequence of local switchings, from T, we get a positive Hushimi tree G_1 with k blocks. This Hushimi tree has the unique cut-vertex v with H-degree k. Let B_1 be a complete block with vertices $a_1, a_2, \cdots, a_\ell, v$. By local switching with respect to $(v, J = B_1 \backslash \{v\})$, from G_1 we get a signed graph G_2. By local switching with respect to $(a_i, J = B_1 \backslash \{a_i\})$, we get a positive Hushimi tree G_3 with k blocks. This G_3 has the unique cut-vertex a_i. By a sequence of local switchings, from G_3 we get the desired tree T'.

Lemma 12. Let T_1 and T_2 be line-trees of order n. Then, T_1 is equivalent to T_2 by local switching if and only if T_2 is a permutation of T_1.

Proof. Let T_1 be a line-tree $a_1 a_2 \cdots a_n$ and T_2 be its permutation $a_{i_1} a_{i_2} \cdots a_{i_n}$. Then, T_1 and T_2 are equivalent by local switching to the complete graph with vertices $\{a_1, a_2, \cdots, a_n\}$. Hence, they are equivalent by local switching. Conversely, if a line-tree T_1 is equivalent to a line-tree T_2 by local switching, it is evident that T_2 is a permutation of T_1.

Proof of Theorem 8. Let T_2 be a permutation of T_1. Then using lemmas 11 and 12, we can construct a sequence of local switchings by which T_1 is transformed to T_2. On the other hand, when a tree is transformed to another tree by local switchings, by taking account of lemma 9, we can use local switchings such that are treated in lemmas 11 and 12. Hence we only interchange vertices of the tree.

4. The lattice D_n and signed cycles

A $k-$cycle $C^k = (V, E)$, where $V = \{a_1, a_2, \cdots, a_k\}$, $E = \{a_1a_2, a_2a_3, \cdots, a_{k-1}a_k, a_ka_1\}$, will be denoted simply $C^k = a_1a_2 \cdots a_ka_1$. For signed cycles, there are two switching classes, which are distinguished by the parity or the balance, where the parity of a signed cycle is the parity of the number of its edges which carry a positive sign and the balance is the product of the signs on its edges [?].

The lattice D_n is spanned by vectors $\pm e_i \pm e_j$, $(1 \leq i \neq j \leq n)$, where $\{e_1, \cdots, e_n\}$ is the orthonormal base of the euclidean n-space R^n. There is the one-to -one correspondence between ordered root bases of D_n and connected signed graphs associated with D_n.

Theorem 13. Let C^k be a $k-$cycle. Then, it is equivalent to a tree by local switching if and only if its parity is odd.

Proof. Let the parity be odd. If the parity of k is odd, then by switching, we may assume that signs of all edges are positive. Put $C^k = a_1a_2 \cdots a_ka_1$. By a sequence of local switchings with respect to $(a_2, J = \{a_3\})$, $(a_3, J = \{a_4\})$, $\cdots$, $(a_{k-1}, J = \{a_k\})$, we get a signed graph G, which is the graph obtained from the positive complete graph on vertices $\{a_1, a_2, \cdots, a_k\}$ by deleting the edge a_1a_k. By a sequence of local switchings with respect to $(a_3, J = \{a_2\})$, $(a_4, J = \{a_3\})$, $\cdots$, $(a_{k-1}, J = \{a_{k-2}\})$, from the graph G, we get a tree with edge set $E = \{a_2a_3, a_3a_4, \cdots, a_{k-2}a_{k-1}, a_{k-1}a_k, a_{k-1}a_1\}$, which may be regarded as the Dynkin diagram of D_k.

When the parity of k is even, we get a tree as similarly as above.

Now assume that the parity of C^k is even. For a cycle $a_1a_2a_3a_1$, we may assume that only the edge a_1a_2 has negative sign. Then we can not transform it to a tree by local switching. Next, every edge of a cycle $a_1a_2a_3a_4a_1$ has positive sign. We must transform it by local switching, for example, with respect to $(a_2, \{a_1\})$. Then, we have a signed graph with $E^+ = \{a_1a_2, a_1a_3, a_2a_3, a_4a_1\}$, $E^- = \{a_3a_4\}$. This graph can not be transformed to a tree by local switching. Now suppose that any $k-1$-cycle with even parity can not be transformed to a tree by a sequence of local switching. Take a k-cycle $a_1a_2 \cdots c_kc_1$ with even parity. We must do some local switching, for example, with respect to $(a_1, J = \{a_k\})$. We get a signed graph and its induced cycle $a_2a_3a_4 \cdots a_ka_2$ with even parity. Any local switching of the signed graph at some $a_j, 2 \leq j \leq k$, has the same effect on the induced cycle $a_2a_3a_4 \cdots a_ka_2$ as local switching at a_j of the cycle $a_2a_3a_4 \cdots a_ka_2$. As the cycle $a_2a_3a_4 \cdots a_ka_2$ can not be transformed to a tree, the induced cycle $a_2a_3a_4\cdots a_ka_2$ and hence the k-cycle $a_1a_2\cdots c_kc_1$ can not be transformed to a tree by a sequence of local switchings.

We denote by $[D_k]$ the tree which is isomorphic to the Dynkin diagram of D_k, and by $K_k - e$ the graph obtained from the positive complete graph on k vertices by deleting one edge. In the above proof, we proved already

Theorem 14. Let C^k be a $k-$cycle with odd parity. Then, C_k, $[D_k]$ and $K_k - e$ are equivalent by local switching.

Theorem 15. Any signed graph associated to the lattice D_n is equivalent to the tree $[D_n]$ by local switching.

Proof. Let G be a signed graph corresponding to an ordered base $\{a_1, a_2, \cdots, a_n\}$ of the lattice D_n. If we replace a_i by $-a_i$, then the sign of G is switched with respect to $\{a_i\}$. Hence there is no problem whether we take a_i or $-a_i$. If $a_i = e_j - e_\ell$ (resp. $a_i = e_j + e_\ell$) is contained in the ordered base, $e_j + e_\ell$ (resp. $e_j - e_\ell$) is not contained in it except one pair which we denote by $a_{k-1} = e_{k-1} - e_k, a_k = e_{k-1} + e_k$, $(1 \leq k \leq n$). It leaves the switching class of G invariant to replace $a_i = e_j - e_\ell$ (resp. $a_i = e_j + e_\ell$) by $e_j + e_\ell$ (resp. $e_j - e_\ell$). Hence, we always take $a_i = e_j - e_\ell, (j < \ell)$, if either of $e_j - e_\ell$ or $e_j + e_\ell$ is contained in the ordered base.

If G is a graph corresponding to the base $\{a_1 = e_1 - e_2, a_2 = e_2 - e_3, \cdots, a_{n-1} = e_{n-1} - e_n, a_n = e_{n-1} + e_n\}$, G is just the tree $[D_n]$.

Assume that G is a graph corresponding to the base $\{a_1 = e_1 - e_2,\ a_2 = e_2 - e_3,\ \cdots, a_{k-1} = e_{k-1} - e_k, a_k = e_{k-1} + e_k, a_{k+1} = e_k - e_{k+1}, \cdots,\ a_n = e_{n-1} - e_n\}$, $(2 < k < n)$. Then it is a signed graph with edge sets $E^+ = \{a_1a_2, a_2a_3, a_3a_4, \cdots, a_{k-2}a_{k-1}, a_{k-2}a_k, a_{k-1}a_{k+1}, a_{k+1}a_{k+2}, \cdots, a_{n-1}a_n\}$ and $E^- = \{a_ka_{k+1}\}$. By a sequence of local switchings, from G, we get a signed graph G_1 with three blocks B_1, B_2 and B_3, where B_1 and B_3 are the positive complete graphs on vertices $\{a_1, \cdots, a_{k-2}\}$ and vertices $\{a_{k+1}, \cdots, a_n\}$, and B_2 is a 4-cycle $a_{k-2}a_{k-1}a_{k+1}a_ka_{k-2}$ with odd parity. By local switchings with respect to $(a_{k-2}, J = \{a_{k-1}, a_k\})$, $(a_{k+1}, J = \{a_{k-1}, a_k\})$ and $(a_k, J = B_1)$, from G_1, we get a signed graph which is isomorphic to $K_n - e$.

In general, let G be a signed graph corresponding to an ordered base $\{a_1, a_2,\ \cdots, a_n\}$, where we may assume that $a_{k-1} = e_{k-1} - e_k, a_k = e_{k-1} + e_k$ is the particular pair. By a similar argument as in the proof of theorem 1, we can show that G consists of ℓ blocks $B_1, B_2, \cdots, B_\ell$ such that $B_1, B_2, \cdots, B_{\ell-1}$ are complete blocks and B_ℓ is given by $\{a_{k-1} = e_{k-1} - e_k, a_k = e_{k-1} + e_k, a_{i_1} = e_{k-1} - e_{j_1}, \cdots, a_{i_s} = e_{k-1} - e_{j_s}, a_{u_1} = e_k - e_{v_1}, \cdots, a_{u_t} = e_k - e_{v_t}\}$, where all $e_{k-1}, e_k, e_{j_1}, \cdots, e_{j_s}, e_{v_1}, \cdots, e_{v_t}$ are different. For any cut-vertex a of G, we can show as similarly as in the proof of theorem 1 that $G - a$ has two connected components. By a sequence of local switchings at all cut-vertices, from G, we get a signed graph G_1. If it is necessary, by rearrangement of vertices, G_1 can be expressed as follows. The subgraphs of G_1 on vertices $\{a_1, \cdots, a_{k-2}\}$ and on vertices $\{a_{k+1},\ \cdots, a_n\}$

are complete. Moreover G_1 has the edges $\{a_1a_{k-1},\ a_2a_{k-1}, \cdots, a_{k-2}a_{k-1}, a_1a_k, a_2a_k, \cdots, a_{k-2}a_k, a_{k+1}a_{k-1}, a_{k+2}a_{k-1}, \cdots, a_na_{k-1}\}$ with sign $+1$ and the edges $\{a_{k+1}a_k, a_{k+2}a_k, \cdots, a_na_k\}$ with sign - 1. By local switching with respect to $(a_{k-1}, J = \{a_1, \cdots, a_{k-2}\})$, from G_1, we get $K_n - \{a_{k-1}a_k\}$.

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