Construction of Knut Vik Designs and Orthogonal Knut Vik Designs

By

Toru ISHIHARA

Faculty of Integrated Arts and Sciences,
The University of Tokushima,
Minamijosanjima, Tokushima 770-8502, JAPAN
e-mail address: ishihara@ias.tokushima-u.ac.jp
(Received September 29, 2006)

Abstract

Following Euler's method, A. Hedayat constructs some Knut Vik designs. We call them Knut Vik designs of Hedayat in this note. We give Knut Vik designs of Hedayat explicitly and decide when Knut Vik designs of Hedayat are mutually orthogonal.

2000 Mathematics Subject Classification. 05B15

Introduction

Let $A$ be a Latin square of order $n$, that is, an $n \times n$ array in which $n$ distinct symbols are arranged so that each symbol occurs once in each row and column. Index its rows and columns by $1, 2, \ldots, n$. By the $j$th right diagonal of $A$ we mean the following $n$ cell of $A$:

$$(i, j + i - 1); \quad i = 1, 2, \ldots, n; \quad \text{(mod } n\text{)}$$

We define also the $j$th left diagonal of $A$ to the following $n$ cell of $A$:

$$(i, j - i); \quad i = 1, 2, \ldots, n; \quad \text{(mod } n\text{)}$$

Let $\Sigma$ be a set of $n$ distinct symbols. If we can fill the cells of $A$ by the elements of $\Sigma$ in such a way that each row, column, right diagonal and left diagonal of $A$ contains all the elements of $\Sigma$, we say the resulting structure a Knut Vik design, which we denote by $K$. It is also called a pandiagonal Latin square [1], [3]. In this paper, we set $\Sigma = \{0, 1, 2, \ldots, n - 1\}$. It is well known that Knut Vik designs of order $n$ exists if and only if $n$ is not divisible by 2 or 3. A. Hedayat [3] showed that $n$ is not divisible by 2 or 3 then $K = (k_{ij})$ with $k_{ij} = \lambda i + j (\text{mod } n)$ is a Knut Vik design if $\lambda, \lambda - 1, \lambda + 1$ are relatively prime.
to $n$. In this paper, we call these Knut Vik designs as Knut Vik designs of Hedayat, Let $n$ have the prime decompsition

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}.$$ 

Then also he showed that there are

$$N = p_1^{a_1-1} p_2^{a_2-1} \cdots p_r^{a_r-1} (p_1 - 3)(p_2 - 3) \cdots (p_r - 3).$$

different choices for $\lambda$. In the present note, we define a standard way which gives $\lambda$ satisfying the condition that $\lambda, \lambda - 1, \lambda + 1$ are relatively prime to $n$. K. Afsarinejad showed that there exist at least $\min(p_i - 3), \ (i = 1, 2, \cdots, r)$ mutually orthogonal Knut Vik designs of order $n$. We show that there exist at most $\min(p_i - 3), \ (i = 1, 2, \cdots, r)$ mutually orthogonal Knut Vik designs of Hedayat. We also obtain that to each Knut Vik design of Hedayat, there are

$$p_1^{a_1-1} p_2^{a_2-1} \cdots p_r^{a_r-1} (p_1 - 4)(p_2 - 4) \cdots (p_r - 4)$$

orthogonal Knut Vik designs of Hedayat.

1. A standard construction of Knut Vik designs of Hedayat

Let $n$ be not divisible by 2 or 3 and have the prime decomposition

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}, \quad (3 < p_1 < p_2 < \cdots < p_r).$$

If $\lambda, \lambda + 1, \lambda - 1$ are relatively prime to $n$, then

$$K = (k_{ij}) \quad \text{with} \quad k_{ij} = \lambda i + j \pmod{n}$$

is a Knut Vik design of Hedayat. We decide explicitly when $\lambda, \lambda + 1, \lambda - 1$ are relatively prime to $n$.

Put

$$m_i^{a_i} = i_a + 1, \quad 1 \leq a \leq r, \ 1 \leq i_a \leq p_a - 3.$$ 

From Chinese remainder theorem, we obtain

**Lemma 1.** For each $\{i_1, i_2, \cdots, i_r\}$ with $1 \leq i_1 \leq p_1 - 3, 1 \leq i_2 \leq p_2 - 3, \cdots, 1 \leq i_r \leq p_r - 3$, there is a positive integer $m$ satisfies

$$m = m_i^{a_i} \pmod{p_1}, m = m_i^{a_i} \pmod{p_2}, \cdots, m = m_i^{a_i} \pmod{p_r}.$$
As this integer is unique on \( Z(p_1p_2 \cdots P_r) \), we denote it by \( m_{i_1i_2 \cdots i_r} \).

Now, we obtain

**Theorem 2** Set \( n_1 = p_1^{a_1-1}p_2^{a_2-1} \cdots p_r^{a_r-1} \). For each \( \{t, i_1, i_2, \ldots, i_r\} \) with \( 0 \leq t \leq n_1 -1, 1 \leq i_1 \leq p_1 -3, 1 \leq i_2 \leq p_2 -3, \ldots, 1 \leq i_r \leq p_r -3 \), put

\[
\lambda(t, i_1, i_2, \ldots, i_r) = p_1p_2 \cdots p_r t + m_{i_1i_2 \cdots i_r},
\]

then these \( \lambda \)'s give \( N \) different choices for Knut Vik designs of Hedayat.

Following the proof of Chinese remainder theorem, we construct explicitly integers \( m_{i_1i_2 \cdots i_r} \) as follows. Put

\[
q_1 = p_1^{-1} (\mod P_2), q_2 = (p_1p_2)^{-1} (\mod P_3), \ldots, q_{r-1} = (p_1p_2 \cdots p_{r-1})^{-1} (\mod P_r)
\]

Now we get inductively,

\[
\begin{align*}
m_{i_1i_2 \cdots i_r} &= m_{i_1}^1 (\mod p_1), \\
m_{i_1i_2 \cdots i_r} &= m_{i_1}^1 + p_1s_1 = m_{i_2}^2 (\mod p_2), \\
s_1 &= q_1(m_{i_2}^2 - m_{i_1}^1) + s_2p_2, \\
m_{i_1i_2} &= m_{i_1}^1 + p_1q_1(m_{i_2}^2 - m_{i_1}^1), \\
m_{i_1i_2 \cdots i_r} &= m_{i_1i_2} + p_1p_2s_2 = m_{i_3}^3 (\mod p_3), \\
s_2 &= q_2(m_{i_3}^3 - m_{i_1i_2}) + s_3p_3, \\
m_{i_1i_2i_3} &= m_{i_1i_2} + p_1p_2q_2(m_{i_3}^3 - m_{i_1i_2}), \\
m_{i_1i_2 \cdots i_r} &= m_{i_1i_2i_3} + p_1p_2p_3s_3 = m_{i_4}^4 (\mod p_4), \\
&\vdots
\end{align*}
\]

\[
\begin{align*}
m_{i_1i_2 \cdots i_{r-1}} &= m_{i_1i_2 \cdots i_{r-2}} + p_1p_2 \cdots p_{r-2}q_{r-2}(m_{i_{r-1}}^{r-1} - m_{i_1i_2 \cdots i_{r-2}}), \\
m_{i_1i_2 \cdots i_r} &= m_{i_1i_2 \cdots i_{r-1}} + p_1p_2 \cdots p_{r-1}s_{r-1} = m_{i_r}^r (\mod p_r), \\
s_{r-1} &= q_{r-1}(m_{i_r}^r - m_{i_1i_2 \cdots i_{r-1}}) (\mod (p_1p_2 \cdots p_r)), \\
m_{i_1i_2 \cdots i_r} &= m_{i_1i_2 \cdots i_{r-1}} + p_1p_2 \cdots p_{r-1}q_{r-1}(m_{i_r}^r - m_{i_1i_2 \cdots i_{r-1}})m_{i_1i_2 \cdots i_{r-1}} (\mod (p_1p_2 \cdots p_r)).
\end{align*}
\]

**Example 1.** Let \( n = 5 \times 7 \). Then \( p_1 = 5, p_2 = 7 \) and \( q_1 = 3 (\mod 7) \). We have

\[
m_1^1 = 2, m_2^2 = 3, m_1^2 = 2, m_2^3 = 3, m_3^2 = 4, m_4^2 = 5.
\]

Hence, we get

\[
\begin{align*}
m_{i_1i_2} &= m_{i_1}^1 + 15(m_{i_2}^2 - m_{i_1}^1) (\mod 35), \\
&= i_1 + 1 + 15(i_2 - i_1) (\mod 35), \quad 1 \leq i_1 \leq 2, \quad 1 \leq i_2 \leq 4.
\end{align*}
\]
Thus, we obtain

$$m_{11} = 2, m_{12} = 17, m_{13} = 32, m_{14} = 12, m_{21} = 23, m_{22} = 3, m_{23} = 18, m_{24} = 33 \text{ (mod 35).}$$

**Example 2.** Let \( n = 5 \times 7 \times 11 \). Then \( p_1 = 5, p_2 = 7, p_3 = 11 \), \( q_1 = 3 \) (mod 7), \( q_2 = 6 \) (mod 11). \( m^{1}_{i_1}, m^{2}_{i_2} \) are the same as in Example 1, and \( m^{3}_{i_3} = i_3 + 1, \ 1 \leq i_3 \leq 8 \). It is evident that \( m_{i_1i_2} \ 1 \leq i_1 \leq 2, \ 1 \leq i_2 \leq 4 \) are also the same as in Example 1. Now we have

$$m_{i_1i_2i_3} = m_{i_1i_2} + 210(m^{3}_{i_3} - m_{i_1i_2}) \text{ (mod 385).}$$

Now we write simply \( m_{i_1i_2} \) for \((i_{i_1i_21}, i_{i_1i_22}, \ldots, i_{i_1i_28})\).

$$m^{1*}_{11} = (2, 212, 37, 247, 72, 282, 107, 317), \ m^{1*}_{12} = (332, 157, 367, 192, 17, 227, 52, 262),$$

$$m^{1*}_{13} = (277, 102, 312, 137, 347, 172, 382, 207), \ m^{1*}_{14} = (222, 47, 257, 82, 292, 117, 327, 152),$$

$$m^{1*}_{21} = (233, 58, 268, 93, 303, 128, 338, 163), \ m^{1*}_{22} = (178, 3, 213, 38, 248, 73, 283, 108),$$

$$m^{1*}_{23} = (123, 333, 158, 368, 193, 18, 228, 53), \ m^{1*}_{24} = (68, 278, 103, 313, 138, 348, 173, 383).$$

**Example 3.** Let \( n = 7^2 \times 11 \). Then \( p_1 = 7, p_2 = 11, q_1 = 8 \) (mod 11). \( m^{1}_{i_1} = i_1 + 1, m^{2}_{i_2} = i_2 + 1, \ 1 \leq i_1 \leq 4, \ 1 \leq i_2 \leq 8 \). Hence, we get

$$m_{i_1i_2} = m^{1}_{i_1} + 56(m^{2}_{i_2} - m^{1}_{i_1}) = i_1 + 1 + 56(i_2 - i_1) \text{ (mod 77),} \ 1 \leq i_1 \leq 4, \ 1 \leq i_2 \leq 8.$$  

We write \( m^{*}_{i_1} \) for \((i_{i_11}, i_{i_12}, \ldots, i_{i_18})\).

$$m^{*}_{1} = (2, 58, 37, 16, 72, 51, 30, 9), \ m^{*}_{2} = (24, 3, 59, 38, 17, 73, 52, 31),$$

$$m^{*}_{3} = (46, 25, 4, 60, 39, 18, 74, 53), \ m^{*}_{4} = (68, 47, 26, 5, 61, 40, 19, 75).$$

Now we obtain

$$\lambda(t, i_1, i_2) = 77t + m_{i_1i_2}, \ 0 \leq t \leq 6, \ 1 \leq i_1 \leq 4, \ 1 \leq i_2 \leq 8.$$
2. Orthogonal Knut Vik designs of Hedayat

Let $K_1$ and $K_2$ be Knut Vik designs of Hedayat of order $n$. $K_1$ and $K_2$ are said to be orthogonal if they are orthogonal in the sense of Latin squares. Using the notations in Theorem 2, assume that $K_1 = (k_{ij}^{(1)})$ and $K_2 = (k_{ij}^{(2)})$ are given by

$$k_{ij}^{(1)} = \lambda(t_1, i_1^{(1)}, i_2^{(1)}, \ldots, i_r^{(1)})i + j, \quad k_{ij}^{(2)} = \lambda(t_2, i_1^{(2)}, i_2^{(2)}, \ldots, i_r^{(2)})i + j,$$

where

$$0 \leq t_1 \leq n_1 - 1, 1 \leq i_1^{(1)} \leq p_1 - 3, 1 \leq i_2^{(1)} \leq p_2 - 3, \ldots, 1 \leq i_r^{(1)} \leq p_r - 3,$$

$$0 \leq t_2 \leq n_2 - 1, 1 \leq i_1^{(2)} \leq p_1 - 3, 1 \leq i_2^{(2)} \leq p_2 - 3, \ldots, 1 \leq i_r^{(2)} \leq p_r - 3.$$

It is known that $K_1$ and $K_2$ are orthogonal if and only if

$$\lambda(t_1, i_1^{(1)}, i_2^{(1)}, \ldots, i_r^{(1)}) - \lambda(t_2, i_1^{(2)}, i_2^{(2)}, \ldots, i_r^{(2)}) = \text{and } n \text{ are relatively prime.}

Hence we have

**Lemma 3.** Knut Vik designs of Hedayat $K_1$ and $K_2$ are orthogonal if and only if

$$i_1^{(1)} \neq i_1^{(2)}, i_2^{(1)} \neq i_2^{(2)}, \ldots, i_r^{(1)} \neq i_r^{(2)}.$$

When $K_1$ and $K_2$ are orthogonal, we call $(K_1, K_2)$ an orthogonal pair in this note. From Lemma 3, it follows

**Theorem 4.** For each Knut Vik design of Hedayat, there are

$$p_1^{p_1 - 1}p_2^{p_2 - 1} \cdots p_r^{p_r - 1}(p_1 - 4)(p_2 - 4) \cdots (p_r - 4)$$

Knut Vik designs orthogonal to it. There are

$$\frac{1}{2}p_1^{2p_1 - 2}p_2^{2p_2 - 2} \cdots p_r^{2p_r - 2}(p_1 - 3)(p_1 - 4)(p_2 - 3)(p_2 - 4) \cdots (p_r - 3)(p_r - 4)$$

orthogonal pairs of Knut Vik designs of Hedayat.

Let $S$ be a set of Knut Vik designs of order $n$. We say $S$ to be a set of mutually orthogonal Knut Vik designs of order $n$ if any two Knut Vik designs in $S$ are orthogonal. It is shown by K. Afsarinejad that there are at least $p_1 - 3$ mutually orthogonal Knut Vik designs of order $n$. On the other hand, by Lemma 3, there are at most $p_1 - 3$ mutually orthogonal Knut Vik designs of Hedayat. In fact, $p_1 - 3$ mutually orthogonal Knut Vik designs are given by

$$K_1 = (k_{ij}^{(1)}), K_2 = (k_{ij}^{(2)}), \ldots, K_{p_1 - 3} = (k_{ij}^{(p_1 - 3)}),$$

where
Let $n = 5 \times 7 \times 11$. Then, the maximum possible number of mutually orthogonal Knut Vik designs of Hedayat is 2. There are 672 orthogonal pairs in this case. The maximum possible number of mutually orthogonal Knut Vik designs of Hedayat is also 2.

Let $n = 7^2 \times 11$. In this case, the maximum possible number of mutually orthogonal Knut Vik designs of Hedayat is 4. There are $2^4 \times 3 \times 7^3$ orthogonal pairs and $2^4 \times 3 \times 5 \times 7^5$ sets of 4 mutually orthogonal Knut Vik designs of Hedayat.
References

